# 36 Years of AI and Math
# My Story

**Temporal Reasoning, Graphs and Algorithms**

Martin Charles Golumbic
University of Haifa

**Michal's Question** (my daughter's childhood friend and her sister-in-law):

*In this new world of AI*, **what should I tell my 9th grade son <u>to study</u>?**

My answer:

# Basketball

Learn
from
Basketball!

# Why learn from Basketball ?

- Building **foundational skills** through practice,
- Analyzing data to make **decisions**,
- Applying spatial **awareness to solve problems,**
- Hands-on attempts to achieve a **"solution"**

  *making a shot  =  solving an equation*

- Shooting 100 free throws builds **muscle memory**

  solving 100 equations → conceptual understanding

- **Trying – failing – trying again**

  analyzing why a shot missed  =  why a formula was wrong,

  and adjusting accordingly

# Bottom Line:

Study Mathematics
Study Physics
Keep Programming
Pick a Hobby
Volunteer to Help People – Older and Younger

- **Learn Angles and Trajectory**
- **Learn Geometry and shapes and spaces**
- **Learn about spacing and interaction of players/agents**

**VALUABLE SKILLS FOR THE FUTURE –**

**no matter what**

**Bruno’s question:**

*What could be the content of the talk by Marty Golumbic?*

***What is his “Story”?***

- The early history of interaction between
  Mathematics and Artificial Intelligence
- The development of the idea over the years
- **What are his feelings when he sees the development now?**

## Marty’s answer:

- **AI’s accomplishments** are far beyond the imaginable, but
- **Math’s accomplishments** have also been far beyond the imaginable.

But we HAVE imagined this for many decades!

Just rewatch the movies –

***2001: A Space Odyssey*** **(1968)**
Stanley Kubrick

**Bicentennial Man (1999)**
Robin Williams

**A.I. Artificial Intelligence (2001)**
Steven Spielberg

# John McCarthy quotations

- In 1956, he defined “Artificial Intelligence” as

  “The science of making machines do things that would require intelligence

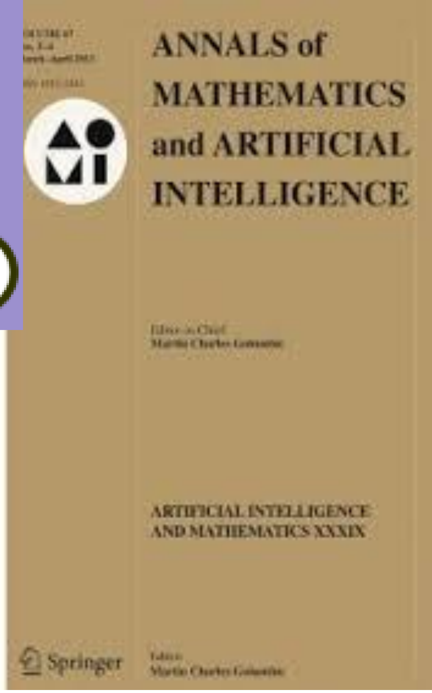
  **if done by people**”
- “As soon as it works, no one calls it AI anymore.”
- “The real problem is not whether machines think, but whether men do.”

**Reflecting the norms of that era.**

The playing field has since evolved,
with prominent participation by women,
a profound and positive impact
on the advancement of research.

# Marty Golumbic quotations

- As the disciplines of AI have **evolved over many decades**, mathematical methods have provided **the solid foundation** of these diverse areas.
- **AI challenges have inspired new Mathematics to meet those challenges.**




- This is why in 1990, we founded the journal ***Annals of Math and AI*** (AMAI) and the biennial ***Intl Symposium on AI and Mathematics*** (ISAIM)
- To foster new areas of applied math and strengthen the scientific underpinnings of AI
- Bridge the gap between new technology applications and foundational mathematical research.
- It was the brainchild of ***Peter L. Hammer*** – to provide a high quality publication outlet for research that was **not (yet) mainstream** – neither in AI nor in Math.

## The range of topics appearing in ISAIM and AMAI has been impressive

search and constraint satisfiability,
AI and statistics,
motion planning in robotics,
data and knowledge bases,
genetic algorithms,
logic and combinatorics,
mathematics of language,
theorem proving,
logic programming,
non-classical logics,
geometric reasoning,
pattern recognition,
formal methods in shape analysis,
boundaries of tractability,
symbolic mathematics,
temporal reasoning,
strategies in automated deduction,
ant robotics,
representations of uncertainty,
multi-agent systems,
discovery and learning,
commonsense reasoning,
concept lattices and their applications,
learning and intelligent optimization, scalable uncertainty management, reactive semantics algorithms, computational social choice,
formal verification and AI,
belief change and probabilistic prediction,
answer set programming,
cognition and neurocomputation,
data science meets optimization,
coalition formation games.

All three of our RISC hosts, Temur, Bruno and Zoltán
have been involved with special issues of AMAI,
including Bruno's 2023 essay,
*Automated programming, symbolic computation, machine learning: my personal view*

And as far back as the 1990s, Jacques Calmet and John Campbell
who edited special issues of AMAI wrote an introductory paper,
*A perspective on symbolic mathematical computing and artificial intelligence*

In fact, I was an invited plenary speaker (Karlsruhe, Germany, Aug. 1992),
at their *First International Conf. on Artificial Intelligence and Symbolic Mathematical Computation.*

***And now to answer –***

***What is Marty's "Story"?***

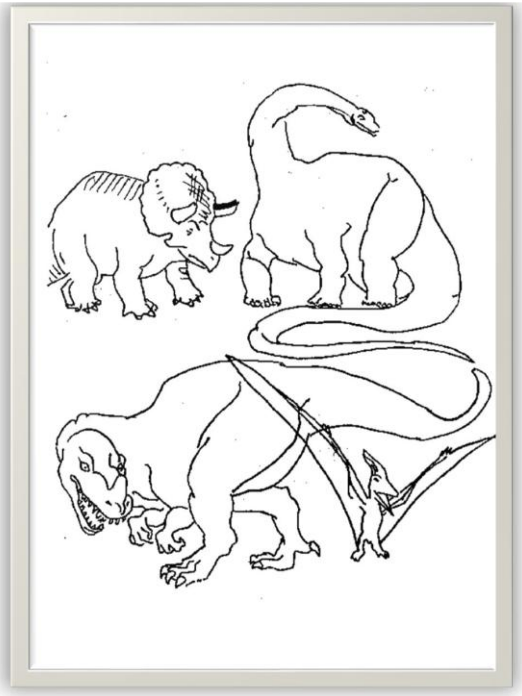
## THE DINOSAURS

*Temporal reasoning is an old science!*

Perhaps not as old as the dinosaurs,

but going back at least to the first paleontologist

looking at a pile of dinosaur bones.




He wonders,

“*What did the dinosaur really look like?*”

“When may they have existed relative to other events?”

“How did they first appear?”

The paleontologist has some *<u>partial information</u>*

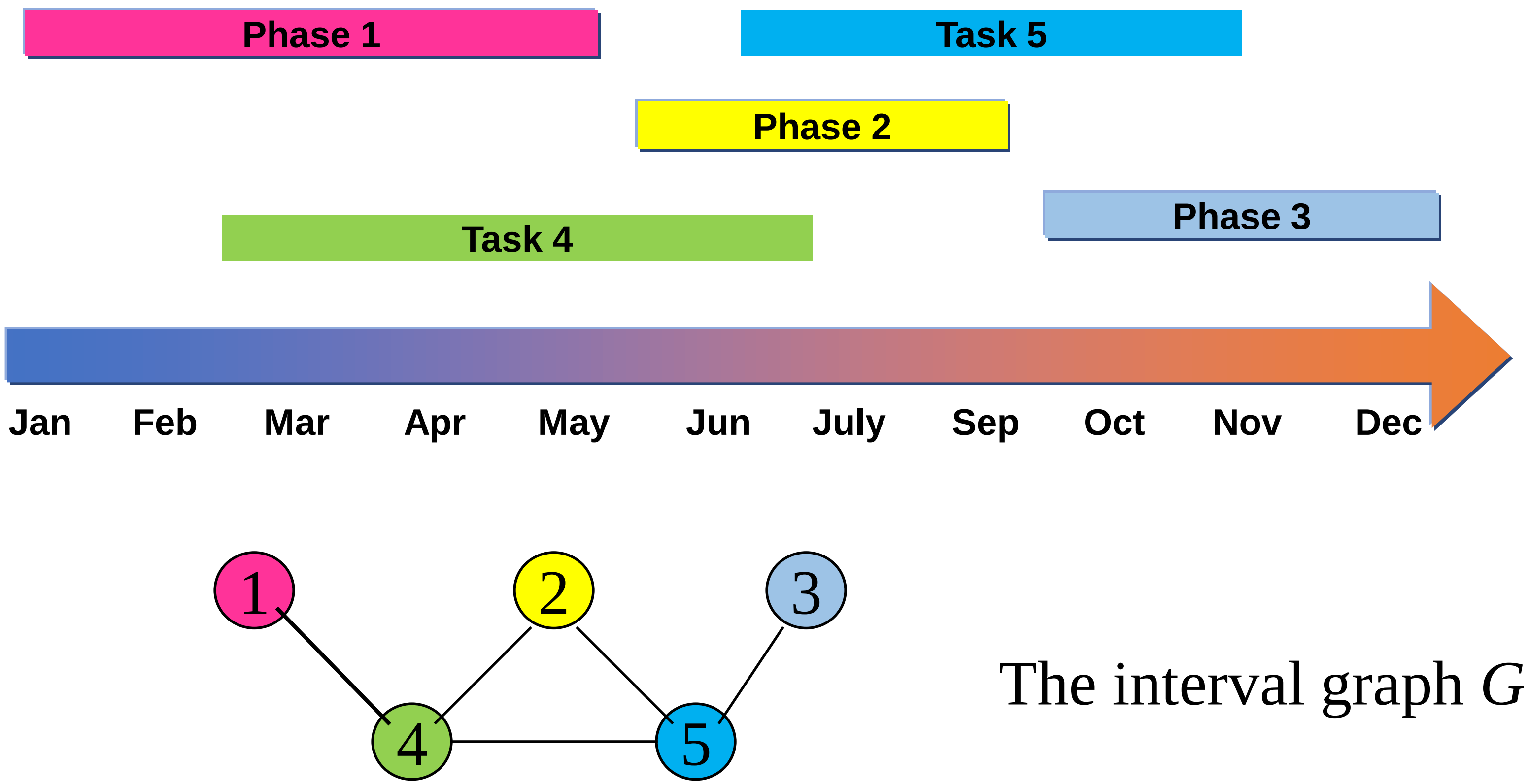
and attempts to *<u>reconstruct</u>* shape and a timeline.

**Can <u>interval</u> <u>graphs</u> help him?**

Today's story: **Temporal Reasoning, Graphs and Algorithms**

## What are interval graphs, and why are they important?

- The most basic model for reasoning about time.
- The **intersection graphs** of intervals on a line.




The interval graph $G$

György Hajós
1912 – 1972

**1957: Each interval corresponds to a vertex of the graph *G*, and two vertices are connected by an edge if and only if the corresponding intervals overlap at least partially.**

Their properties led to Claude Berge to defining and studying "**Perfect Graphs**" starting in the 1960s.

MAXCLIQUE = MINCOLOR
For all induced subgraphs.

**Which guy is Claude Berge?**

Beard or Mustache?

**Theorem.** $G$ is an interval graph if and only if $G$ is $C_4$-free and is a co-comparability graph.

**Theorem.** $G$ is an interval graph if and only if $G$ is chordal and is asteroidal-triple -free.

**Paul Carl Gilmore**
1925 – 2015

**Alan Jerome Hoffman**
1924 – 2021

**Alain Ghouila-Houri**
1939 – 1966

**Cornelis Gerrit Lekkerkerker**
1922 – 1999

**Johan Christoph Boland**
1914 – 1984

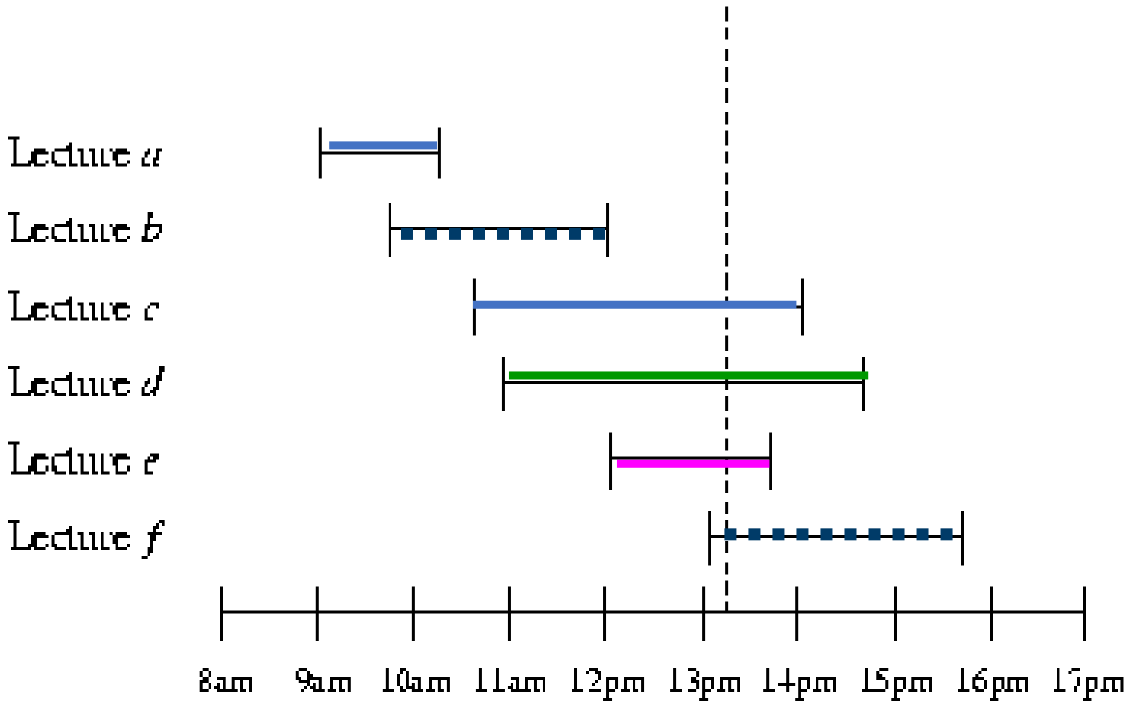
# Applications of Interval Graphs

Genetics Scheduling Artificial Intelligence

VLSI design Computer storage

Frequency assignment Many others …

**Example: Assigning classrooms to Lectures**

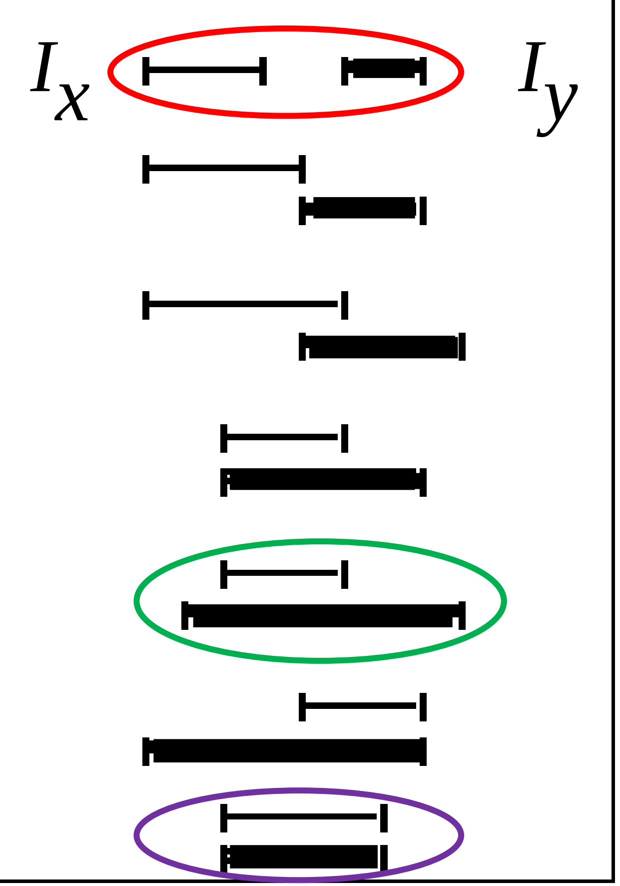



Conflicting lectures → Different rooms

How many rooms?

**Solution: Coloring the vertices of an interval graph**

# James Allen's Temporal Interval Algebra $A_{13}$

| RELATION | NOTATION | INTERPRETATION |
|---|---|---|
| $x$ before $y$ | $\prec$ |  |
| $y$ after $x$ | $\succ$ | |
| $x$ meets $y$ | $m$ | |
| $y$ met-by $x$ | $m^{-1}$ | |
| $x$ overlaps $y$ | $o$ | |
| $y$ overlapped-by $x$ | $o^{-1}$ | |
| $x$ starts $y$ | $s$ | |
| $y$ started-by $x$ | $s^{-1}$ | |
| $x$ during $y$ | $d$ | |
| $y$ includes $x$ | $d^{-1}$ | |
| $x$ finishes $y$ | $f$ | |
| $y$ finished-by $x$ | $f^{-1}$ | |
| $x$ equals $y$ | $\equiv$ | |

Comm. ACM 1983

The **thin** line is the interval $I_x$

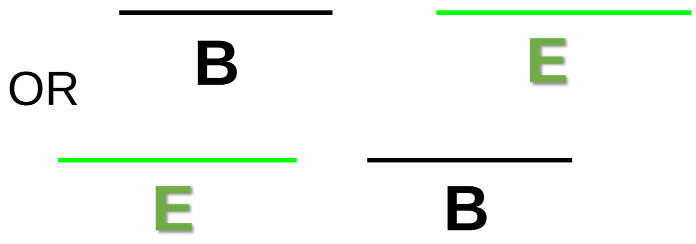
The **thick** line is the interval $I_y$

**Disjunctions:** $x\{\prec, d^{-1}, \equiv\}y$ **means**

**EITHER** $I_x$ ends *before* $I_y$ starts **OR** $I_y$ includes $I_x$ **OR** $I_x$ equals $I_y$

# Examples of Disjunctions

B and E did not exist simultaneously

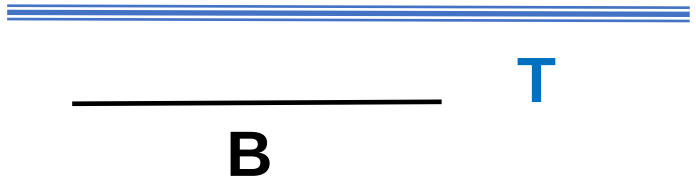



(a) Disjoint intervals
*Two possibilities*

E perished before S emerged

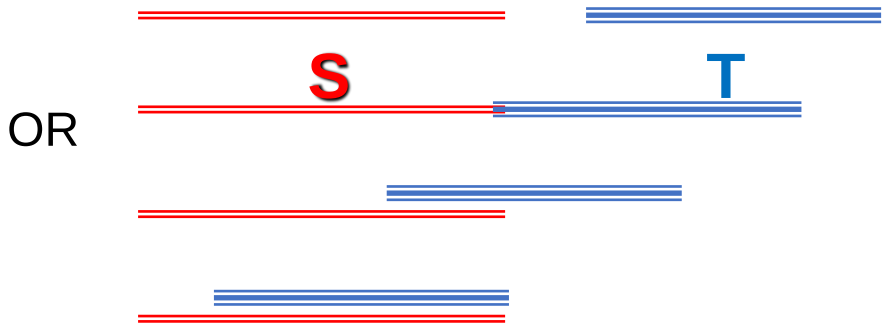

(b) Disjoint and Before
*One possibility*

B emerged and perished during T's time




(c) Containment
*One possibility*

S emerged before T
and perished not after it




(d) Disjoint, touching, overlap, finishing intervals
*Four possibilities*

Either E perished before T emerged
or E emerged when T perished


(e) Before or touching intervals
*Two possibilities*

# Why is it called an Algebra?

Allen [1983] gives the full 13 × 13 composition table for all pairs of the 13 basic interval relations.

Together with an **composition operation**

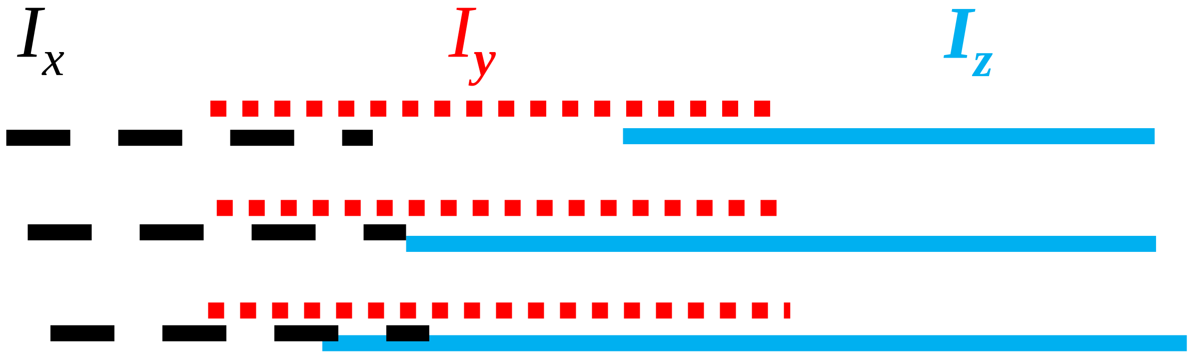
these basic interval relations form a Boolean algebra

a relation algebra in the sense defined by Tarski.

$\prec \bullet \prec \;\Rightarrow\; \prec$ is the transitivity rule: if $I_x \prec I_y \prec I_z$ then $I_x \prec I_z$

$\boldsymbol{m} \bullet \boldsymbol{m} \;\Rightarrow\; \prec$ if $I_x$ meets $I_y$ meets $I_z$ then $I_x \prec I_z$ (a time gap between them)

$o \bullet o \;\Rightarrow\; \{\prec, m, o\}$ *Three possibilities:*



# What is its computational complexity?

**ISAT – Interval SATisfiability: Is there a consistent scenario?**

- There are the 8191 possible disjunctions of $A_{13}$
- The computational complexity is NP-complete
  (like most constraint propagation problems)
- But the original proof was very **UGLY** [Vilain and Kautz, 1986; with bug fixed by Van Beek, 1989]

Enter –Martin Golumbic and Ron Shamir

**Temporal Reasoning meets Computational Biology**

**Both on sabbatical in the USA meeting every couple of weeks at Rutgers (1989-1990).**

What we did –

- Simplify the model to get a cleaner NPC proof
- Find tractable subclasses using graph theory
- In the process, generate new algorithmic research problems

(1) Reduce the "granularity" to 3 basic relations:

$A_3$ $\quad \prec, \succ, \cap = \{s, f, d, m, o, \equiv, s^{-1}, f^{-1}, d^{-1}, m^{-1}, o^{-1}\}$

Thus, reducing to 7 disjunctions of $A_3$ temporal relations

$$\prec, \succ, \cap, \prec\cap, \cap\succ, \prec\succ, \prec\cap\succ$$

instead of 8191 disjunctions of $A_{13}$

disjoint

don't know

(2) Introduce **fragments** (subsets of the disjunctions)

| | |
|---|---|
| Interval orders | $\{\prec, \succ, \cap\}$ |
| Interval graphs | $\{\prec\succ, \cap\}$ |

# Complexity of Testing Consistency

## The Interval Satisfiability Problem (ISAT)

- **ISAT is NP-complete for $A_3$**
  **(Golumbic & Shamir, JACM 1993)**

- Corollary:
  ISAT is NP-complete for Allen's Algebra $\mathbf{A_{13}}$
  (Vilain & Kautz, AAAI 1986,
  with bug fixed by Van Beek 1989)

- **How we did it:**

**Interval Graph Sandwich Problem**

# The graph *sandwich problem*

Take your favorite graph property $\Pi$

The $\Pi$ graph *sandwich problem* asks:

Given: - a vertex set $V$

- a mandatory edge set $E^1$

- a larger edge set $E^2$ $(E^1 \subseteq E^2)$

Question: **Is there a (sandwich) graph $G = (V, E)$**

**with $E^1 \subseteq E \subseteq E^2$ such that $G$ satisfies property** $\Pi$ ?

**Optional edges :** $E^0 = E^2 - E^1$

**Forbidden edges:** $E^3 = V \times V - E^2$

**The Interval Graph Sandwich Problem is precisely the fragment:**

ISAT {≺≻, ∩, ≺ ∩ ≻}

disjoint, intersect, don't care

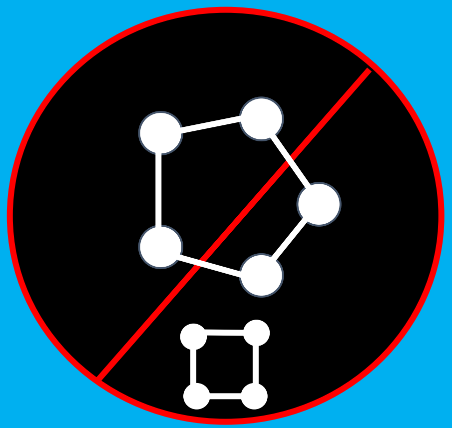
**The Classical Recognition Problem is the case: $E^2 = E^1$**

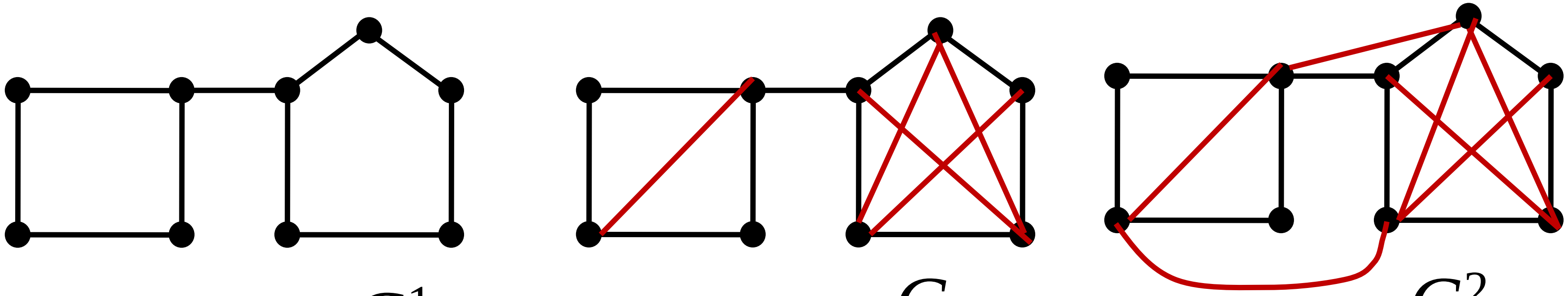
**nothing optional !**

# Example

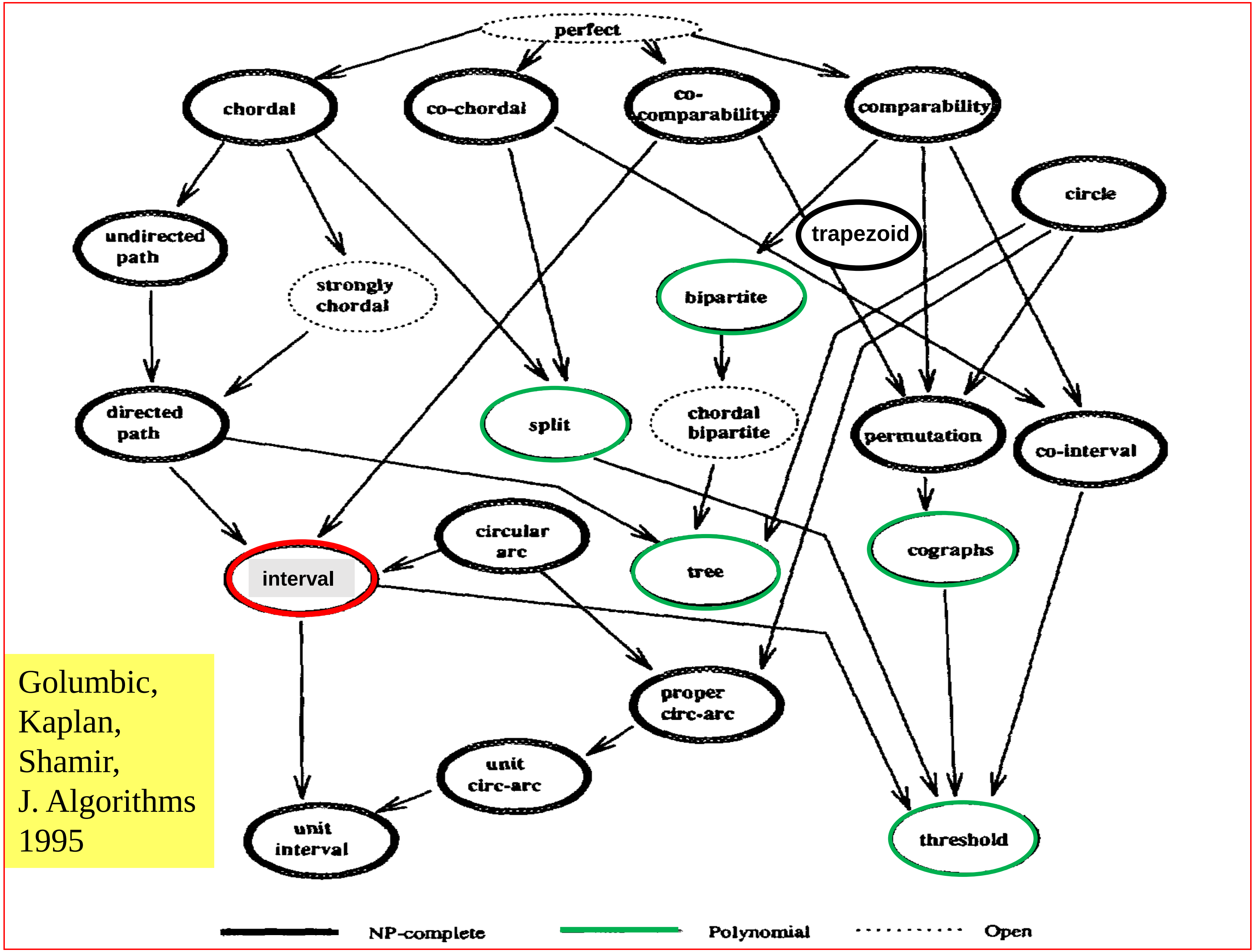
*A Chordal Graph Sandwich*

chordal graph: every cycle of length $\geq$ 4 has a chord





$G^1$ $G$ $G^2$


perfect
chordal
co-chordal
co-comparability
comparability
circle
undirected path
strongly chordal
trapezoid
bipartite
directed path
split
chordal bipartite
permutation
co-interval
circular arc
interval
tree
cographs
proper circ-arc
unit circ-arc
unit interval
threshold
Golumbic, Kaplan, Shamir, J. Algorithms 1995
NP-complete
Polynomial
Open

# Complexity of Testing Fragments of $A_3$

**Minimal Intractable Fragments:**

ISAT is NP-complete even when we have only labels

$\{\prec\succ, \cap, \prec\cap\succ\}$ Golumbic & Shamir, 1993

or $\{\prec\cap, \cap\succ, \prec\succ\}$ Webber, 1995

**Maximal Tractable Fragments:**

ISAT is linear when we have only labels

$\{\prec, \succ, \cap, \prec\cap, \cap\succ, \prec\cap\succ\}$ ← Missing only $\prec\succ$

or $\{\prec, \succ, \prec\cap\succ, \prec\succ\}$

ISAT is $O(n^3)$ when we have only labels

$\{\prec, \succ, \cap, \prec\succ\}$ Golumbic & Shamir, 1993

**WHY IS THIS IMPORTANT?**

# Advantage of Tractable Fragments

Use tractable fragments to

- Test partial solutions and special structured families
- Guide generating choices in global search
- Order variable wisely (as in all CSP)

  including partial look-ahead, etc.

# What about Tractable Subalgebras of $A_{13}$?

Subalgebra: A subset of the disjunctions of Allen's original $A_{13}$ closed under converse, intersection and composition

Theorem (Nebel & Burckert, 1995) The ORD-Horn algebra is the unique maximal tractable algebra containing all the 13 single relations for which ISAT is tractable.

ORD-Horn covers about 11% of $A_{13}$

(does not include $\prec\succ$ )

Are there more?

# More Tractable Subalgebras

Drakengren and Jonsson (1997) found 17 more maximal tractable subalgebras of $A_{13}$

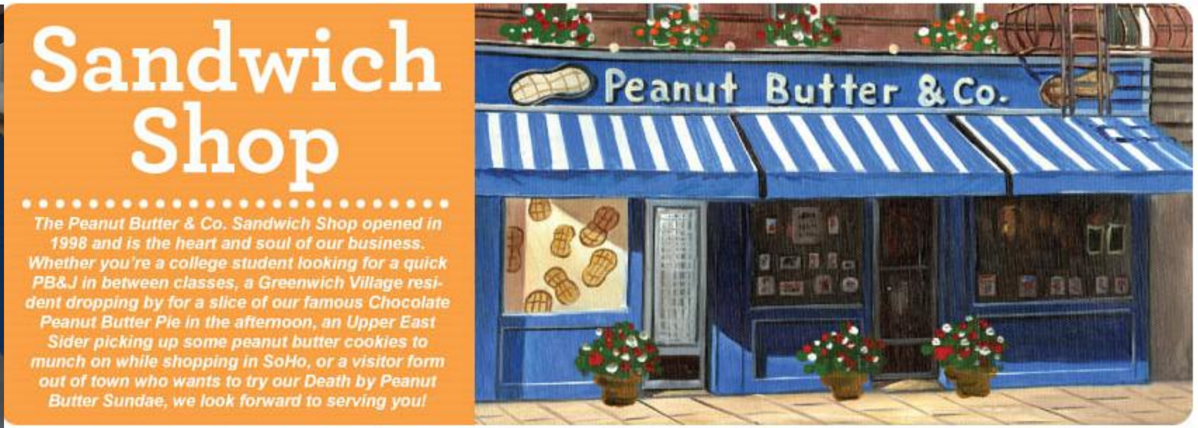
and finally:

Krokhin, Jeavons & Jonsson (2003) gave a complete list of all maximal tractable subalgebras of $A_{13}$

These results all use computer assisted proofs.

# A FEW MORE SANDWICHES BEFORE WE MOVE ON ….

## The Nutropolitan Museum of Art


**The World's First PB&J Art Exhibit!**

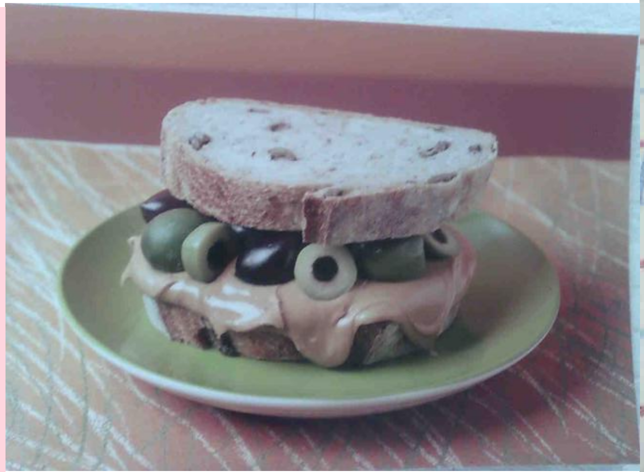
**FREE EVENT - Open to the Public**

**Come by and receive a *FREE* jar of Peanut Butter & Co. peanut butter and we'll *DONATE* a second jar to the Food Bank For New York City in your honor.***

**Just by visiting the gallery you're making a donation!**

**365 PB&J s**tunning photographs one for every day of the year.

**FUN ACTIVITIES FOR EVERYONE!**



# Matrix View of Sandwiches

Adjacency matrix: $\{0, 1, *\}$ entries, where

1 means mandatory edge

0 means required non-edge

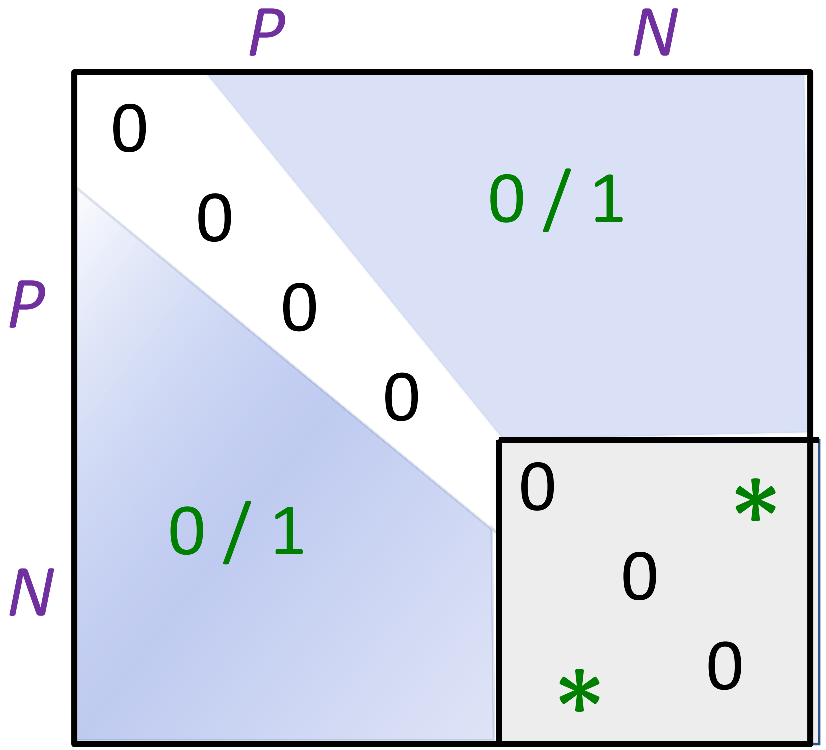
* means optional or don't know or don't care.

$$\begin{bmatrix} 0 & & & & & & & \\ & 0 & & & 0/1/* & & & \\ & & 0 & & & & & \\ & & & 0 & & & & \\ & & & & 0 & & & \\ 0/1/* & & & & & & 0 & \\ & & & & & & & 0 \end{bmatrix}$$

**Can you fill this in to satisfy your graph property $\Pi$?**

# Partitioned Probe Problems

$V(G) = P \cup N$




The Interval Graph **Probe Problem** is Tractable.

# Other Sandwich Problems in Mathematics

- Matrix property sandwich problem  (0/1 completion)
- Hypergraph sandwich problems
- Poset sandwich problems -- and preference relations
- Boolean function completion problems

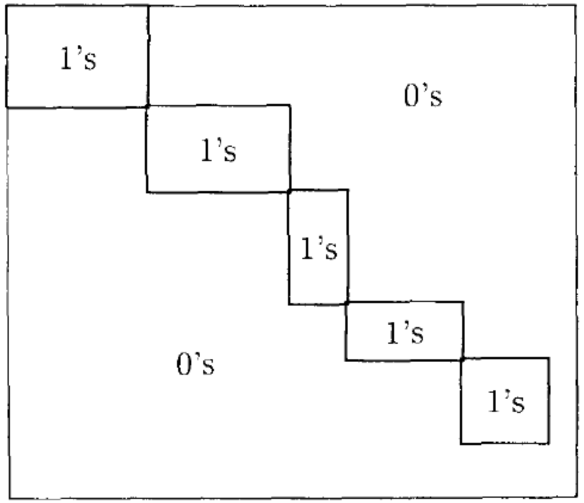



Fig. 1. A rectangular block pattern.

**Logical Analysis of Data (LAD)**

- A powerful technique for data classification
  based on ***partially defined Boolean functions***
- Combines ideas from combinatorics, machine learning and optimization
- Originally introduced by Peter L. Hammer in a lecture in 1986
- Extended and published together with Yves Crama and Toshihide Ibaraki in [1988]

  Cited by 373 publications

# The Theory of Preferences (three good surveys)

Ronen I. Brafman and Carmel Domshlak,
*Preference Handling – An Introductory Tutorial*,
*AI Magazine* 30 (Spring 2009) 58-86.

Carmel Domshlak, Eyke Hüllermeier, Souhila Kaci, Henri Prade,
*Preferences in AI: An overview*,
*Artificial Intelligence* 175 (2011), 1037-1052.

Gabriella Pigozzi, Alexis Tsoukias and Paolo Viappiani,
*Preferences in Artificial Intelligence*,
*Annals Math. and Artif .Intell.* 77 (2016) 361–401.

A **preference** relation $(X, \preceq)$ is a transitive binary relation over $X$.

It is a **total order** if every pair of elements is comparable.

Otherwise, it is a **partial order** (i.e., some outcomes are not comparable).

It is a **strict order** if it is anti-symmetric.

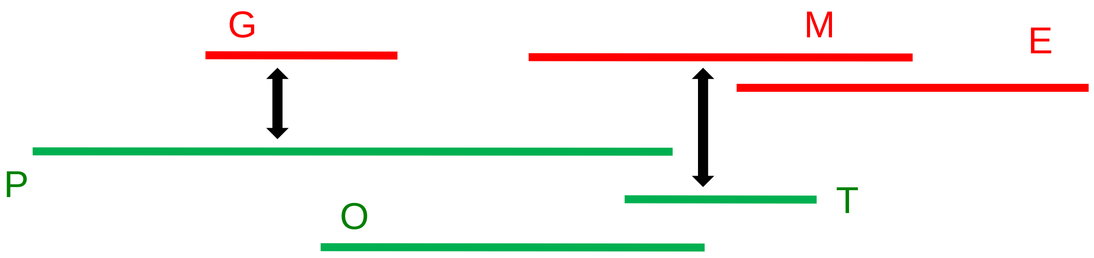
Otherwise it is a **weak order.**

A **weak total order** thus admits a **value function** $h : X \to \mathbb{R}$
where $x \preceq y \Leftrightarrow h(x) \leq h(y)$ and $x \prec y \Leftrightarrow h(x) < h(y)$ .

When a structure is defined by a **vector of $k$ attributes**,
each attribute may order the set $X$ differently, and have its own value function.
These may be independent, conditionally independence, or highly dependent.

# Interval Preference Orders

Gary got between  75 and 80
Mark made between  85 and 95
Elana earned between  90 and 100

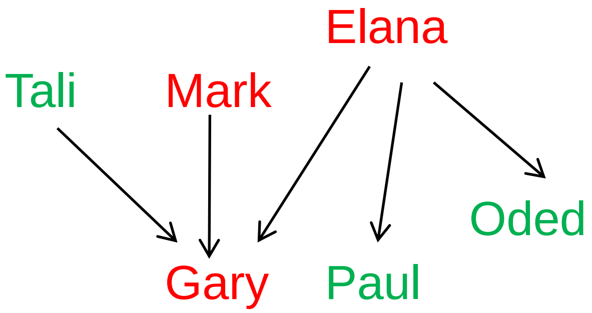



**Let’s add 3 more**

Paul produced between  70 and 88
Tali took between  87 and 93
Oded obtained between  78 and 89

**Should we prefer Elana over Mark?**

Should we prefer Gary over Paul?
Should we prefer Mark over Tali?




The Interval Order

# PQI Interval Orders

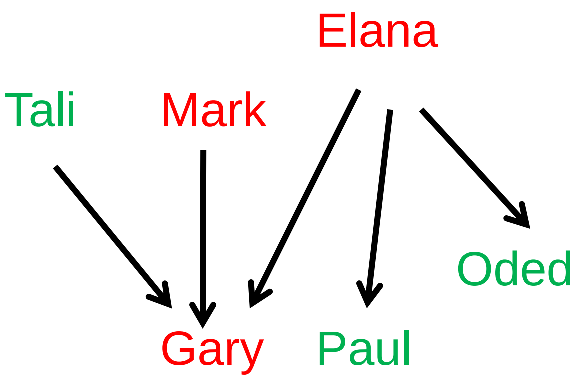
| | | | |
|---|---|---|---|
| P | strict preference | $\prec$ | disjoint intervals |
| Q | weak preference | $\propto$ | overlapping intervals |
| I | indifference | $\oplus$ | containing intervals |

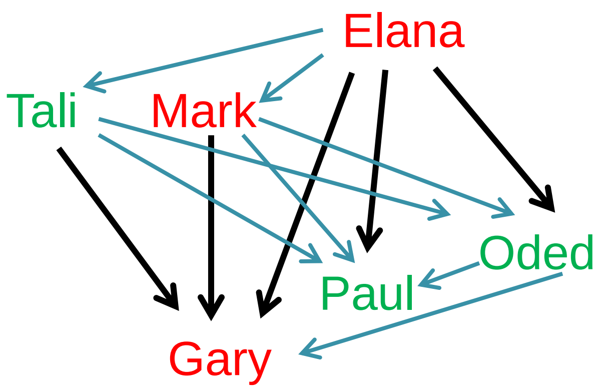
where $\oplus = \{ \subset, \supset, \equiv \}$

The name PQI interval orders is found in Vincke (1988)
but the ideas more generally are already dealt in Allen (1983)
and further developed in Golumbic & Shamir (1993).

Remark: Combining $\prec\propto$ is also a partial order and is equivalent to ordering by “midpoints” of intervals.

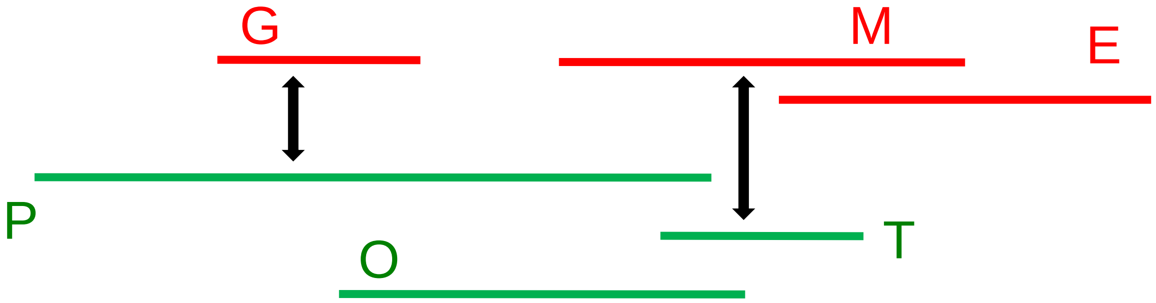
$wd = 1$



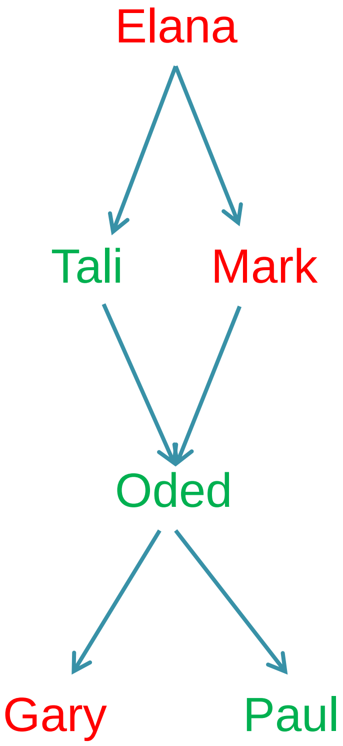




The Interval Order: $\prec$ strict preference

Now add overlap: $\propto$ weak preference



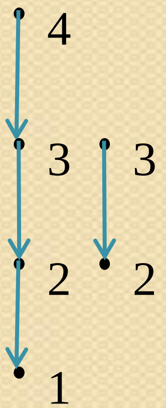

$wd = 0$



# Weak Discrepancy in Partial Orders

Introduced by
Ann Trenk (1998)

**Weak orders are those with** $\boldsymbol{wd = 0}$



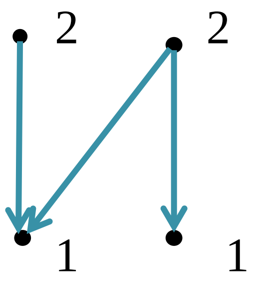

$wd = 2$

## How far is $P = (X, \prec)$ from being a total ranking?

(*a weak order* : a linear ordering of antichains)

weak discrepancy $wd(P)$ : smallest $k \geq 0$ for which there exists an integer ranking function $f : V \rightarrow \mathbf{Z}$ satisfying

(1) if $a \prec b$ then $f(a) + 1 \leq f(b)$ ("up" constraints)
(2) if $a \parallel b$ then $|f(a) - f(b)| \leq k$ ("side" constraints)




$wd = 1$

Motivation: A Story

A manager who partially orders her employees by their value to the company, needs to assign a salary level to each employee.

The "up" constraints ensure that a more valuable employee gets a higher salary than a lesser one.

The "side" constraints are fairness conditions that restrict the salary discrepancies between pairs of incomparable employees.

# Complexity Results

Determining $wd(P)$ and finding an optimal ranking is polynomial-time. (Trenk, 1998)

Moreover,

$$\boldsymbol{wd\,(P) = max_C \lceil \#ups(C) \,/\, \#sides(C) \rceil}$$

taken over all "forcing cycles" $C$ of $P$

(Gimble & Trenk, 1998)

A forcing cycle (preference-indifference cycle):

$[\, a_1, a_2, \ldots, a_m\, , a_1 \,]$ $(m \geq 2)$

such that

$a_i \prec a_{i+1}$ or $a_i \parallel a_{i+1}$ for all $1 \leq i \leq m$

# $\mathbb{PQI}$ Interval Orders $\prec \propto \mathbb{O}$

- $\mathbb{P}$ alone is an interval order
- $\mathbb{Q} \cup \mathbb{Q}^{-1} \cup \mathbb{I}$ is an interval graph

- $\mathbb{P} \cup \mathbb{Q}$ is an order of partial order dim 2
- $\mathbb{I}$ alone is an interval containment graph (equiv to being a permutation graph) and also has partial order dim 2

All of these can be recognized in polynomial time.

# Interval Satisfiability and Preference Orders

## Following the ideas in Allen (1983) and Golumbic & Shamir (1993)

**The 13-valued interval algebra $A_{13}$**

| RELATION | NOTATION | INTERPRETATION |
|---|---|---|
| $x$ before $y$ | $\prec$ | |
| $y$ after $x$ | $\succ$ | |
| $x$ meets $y$ | $m$ | |
| $y$ met-by $x$ | $m^{-1}$ | |
| $x$ overlaps $y$ | $\propto$ | |
| $y$ overlapped-by $x$ | $\propto^{1}$ | |
| $x$ starts $y$ | $s$ | |
| $y$ started-by $x$ | $s^{-1}$ | |
| $x$ during $y$ | $d$ | |
| $y$ includes $x$ | $d^{-1}$ | |
| $x$ finishes $y$ | $f$ | |
| $y$ finished-by $x$ | $f^{-1}$ | |
| $x$ equals $y$ | $\equiv$ | |

$$A_7 : \{\prec, \succ, \alpha, \alpha^{-1}, \subset, \subset^{-1}, \equiv\}$$

Problem Input:
Disjunctions of these seven preference relations for each pair of items

**The macro relation algebra $A_3$**

$$A_3 : \{\prec, \succ, \cap\}$$

## What fragments would be reasonable for $A_7$?

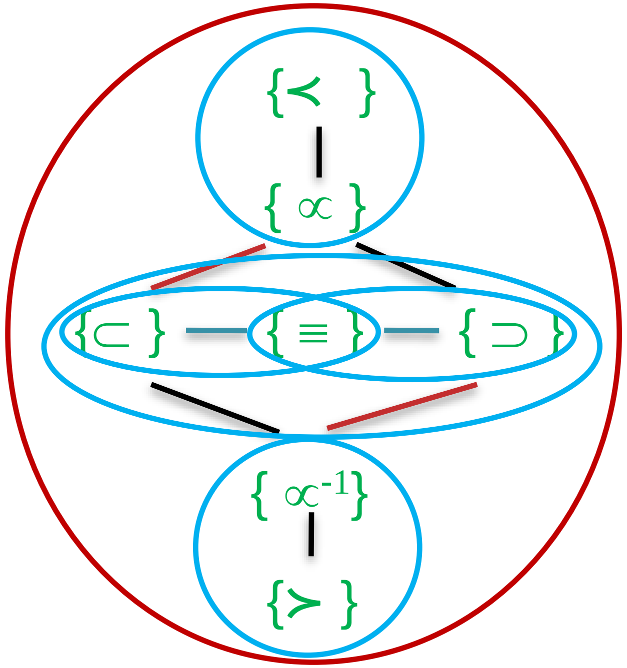
Convex Disjunctions:

$\{\prec\}$ $\{\propto\}$ $\{\subset\}$ $\{\equiv\}$ $\{\supset\}$ $\{\propto^{-1}\}$ $\{\succ\}$ singletons

$\{\prec \propto\}$ $\{\propto^{-1} \succ\}$ $\{\subset \equiv\}$ $\{\equiv \supset\}$ $\{\subset \equiv \supset\}$

$\{\prec \propto \subset \equiv \supset \propto^{-1} \succ\}$ (i.e., don't know/care anything)

And others




Notice that $\{\subset \supset\}$ is NOT here

**On Poset Sandwich Problems**

Michel Habib
David Kelly
Emmanuelle Lebhar
Christophe Paul

April 2003

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# Can transitive orientation make sandwich problems easier?

Michel Habib[a], David Kelly[b], Emmanuelle Lebhar[c,1], Christophe Paul[a,*,1]

The **comparability <u>graph</u>** *sandwich problem*:

Input: *Undirected graphs* $G^1 = (V, E^1)$ , $G^2 = (V, E^2)$ s.t. $E^1 \subseteq E^2$

Question: Is there a **transitively orientable graph** $G = (V, E)$

with $E^1 \subseteq E \subseteq E^2$ ? That is, $E$ has a transitive orientation.

**This problem is NP-Complete.**

The **transitive <u>digraph</u>** *sandwich problem*:

Input: *Directed graphs* $D^1 = (V, F^1)$ , $D^2 = (V, F^2)$ s.t. $F^1 \subseteq F^2$

Question: Is there a **transitive digraph** $D = (V, F)$

with $F^1 \subseteq F \subseteq F^2$ ?

**This problem is EASY !**

Just check if the transitive closure $F^*$ of $F^1$ is contained in $F^2$

So the INTERESTING **poset sandwich problems**

*to be studied must place a* <u>*special property*</u> *on the sandwich F*:

For example,

- *interval order* sandwich problem
- *dimension 2 poset* sandwich problem
- *series-parallel poset* sandwich problem
- *semiorder* sandwich problem
- *lattice* sandwich problem (*not lettuce*)

Moreover, there are <u>two versions</u> of the poset sandwich problem –

<u>depending on the INPUT</u>: a Digraph or a Poset.

**Habib, et al. [2007]:** **Let $\Pi$ be a poset property.**

**DIGRAPH SANDWICH PROBLEM FOR POSET PROPERTY $\Pi$**

*Input*: Two digraphs $D^1 = (V, F^1)$ , $D^2 = (V, F^2)$ such that $F^1 \subseteq F^2$

*Question*: Does there exist a poset $P = (V, F)$ satisfying $\Pi$ with $F^1 \subseteq F \subseteq F^2$ ?

**POSET SANDWICH PROBLEM FOR POSET PROPERTY $\Pi$**

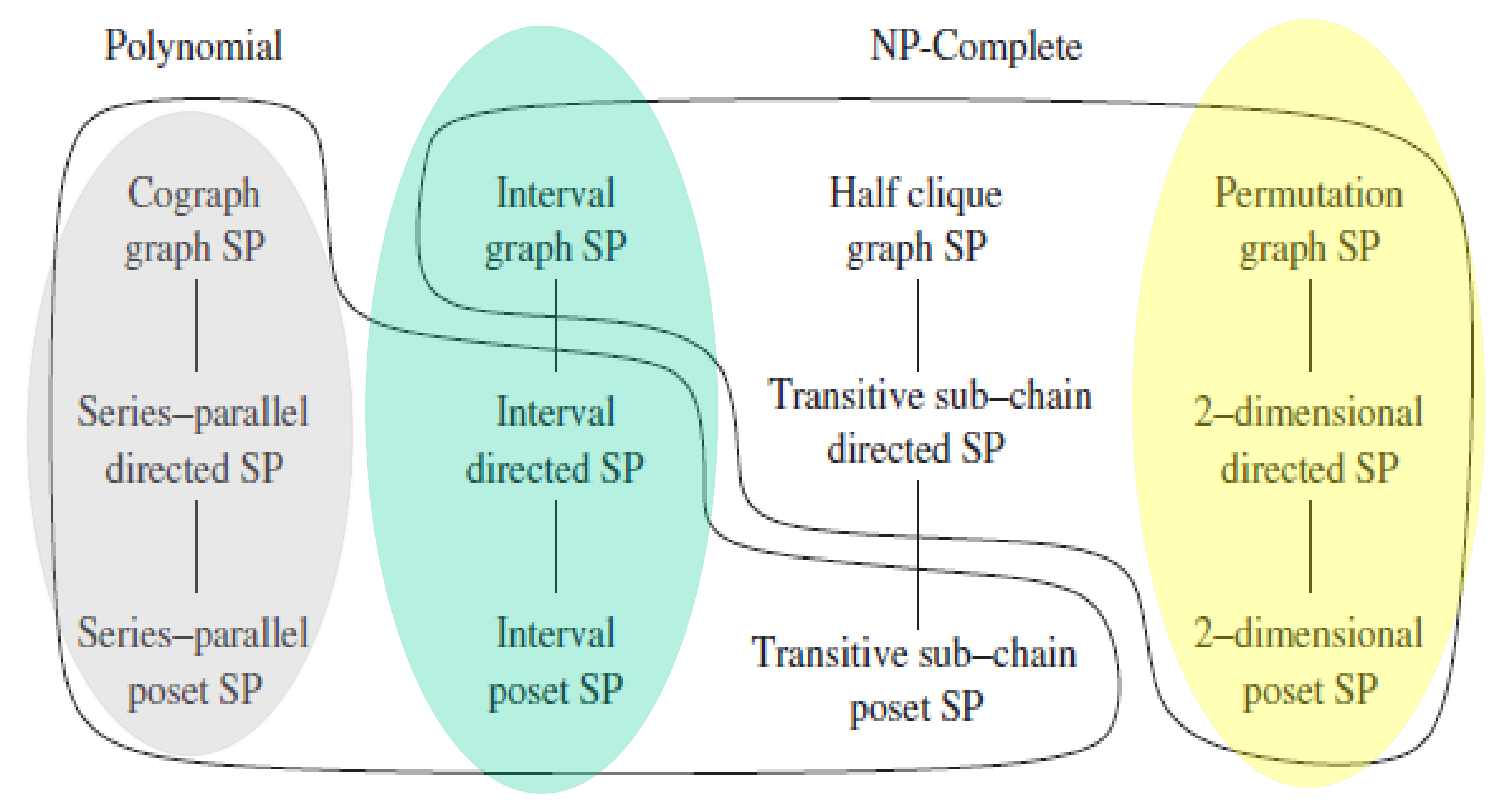
*Input*: Two posets $P^1 = (V, F^1)$ , $P^2 = (V, F^2)$ such that $F^1 \subseteq F^2$

*Question*: Does there exist a poset $P = (V, F)$ satisfying $\Pi$ with $F^1 \subseteq F \subseteq F^2$ ?

**How are they different?**

- In the **DIGRAPH SANDWICH PROBLEM** $F^1$ and $F^2$ *are arbitrary*
- In the **POSET SANDWICH PROBLEM** $F^1$ and $F^2$ *are already transitive*

**Habib, et al. [2007]:**



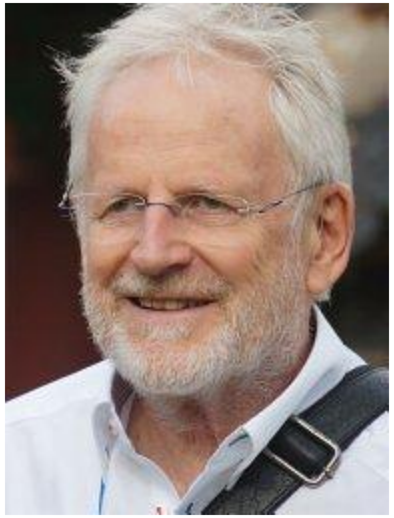
# Conclusions

Sandwich problems are the source of many interesting algorithmic and combinatorial questions.

They are motivated naturally by a number of applications, and are worthy of further research investigation.

**Galileo Galilei** wrote in **1623**:

*"The book of nature is written in the language of mathematics."*

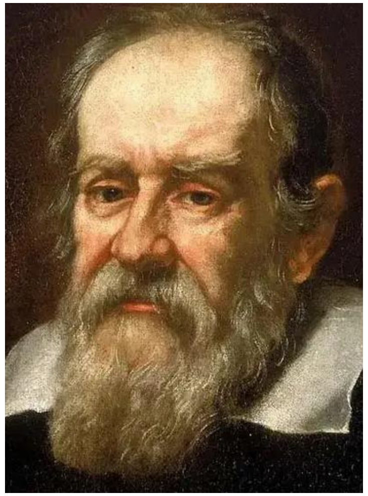


Simply put,

to understand the universe,

to understand nature,

you have to understand mathematics.



**Bruno Buchberger** wrote in **2023**:

*"Symbolic computation and machine learning are fundamentally different,*
they address two completely different ways of how problems are specified.
Together, they constitute (part of) what some people call "artificial intelligence".

"However, **both approaches are just part of algorithmic mathematics.**"

# THANK YOU