Towards Quantum-Reservoir Trajectory Signatures for Symbolic Rewriting Dynamics

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Motivation: symbolic dynamics and feature learning

General question

Can nonlinear feature representations and machine-learning/time-series methods reveal useful structure in symbolic rewriting dynamics?

- Symbolic rewriting is a general model of computation.
- In sufficient generality, reachability and non-reachability questions are undecidable.
- Even in structured algebraic systems, reachability can be very hard to analyse directly.
- Instead of looking only at individual states, we study rewriting *trajectories*.
- Quantum reservoirs provide nonlinear feature maps from symbolic states to continuous spaces.

$$s_0, \ldots, s_T \quad \longmapsto \quad \Phi_{\mathrm{QR}}(s_0), \ldots, \Phi_{\mathrm{QR}}(s_T), \qquad \Phi_{\mathrm{QR}}(s_i) \in \mathbb{R}^d.$$

Symbolic testbed: Andrews–Curtis rewriting

We consider states given by pairs of group words

$$(r_1, r_2) \in F(a, b)^2,$$

where $F(a, b)$ is the free group on generators a, b, with inverses denoted by $A = a^{-1}$ and $B = b^{-1}$.

The rewriting rules, understood modulo free-group reduction, include:

$$\begin{aligned}
\text{inversion:} \quad & (r_1, r_2) \mapsto (r_1^{-1}, r_2), \\
\text{multiplication:} \quad & (r_1, r_2) \mapsto (r_1 r_2, r_2), \\
\text{conjugation:} \quad & (r_1, r_2) \mapsto (g^{-1} r_1 g, r_2),
\end{aligned}$$

where $g \in \{a, A, b, B\}$, together with symmetric variants acting on r_2.

Andrews–Curtis conjecture, 1965

Every balanced presentation of the trivial group is reachable from (a, b) by these transformations.

Why Andrews–Curtis rewriting is hard

A tiny example of rewriting

$$(ab, b) \longmapsto (ab, B) \longmapsto (abB, B) \longmapsto (a, B) \longmapsto (a, b),$$

where $B = b^{-1}$, and words are freely reduced modulo $bB = 1$.

Akbulut–Kirby family

A classical family of potential counterexamples is

$$AK(n) = \left\langle a, b \;\middle|\; a^n b^{-(n+1)},\ abab^{-1}a^{-1}b^{-1} \right\rangle, \qquad n \geq 2.$$

Andrews–Curtis conjecture.

Broader picture

The conjecture remains open. It is widely suspected that it may be false, and families of balanced presentations such as $AK(n)$ have long served as candidate sources of counterexamples.

Full and restricted rewriting systems

- We study the full Andrews–Curtis rule system.
- We also compare it with systems obtained by removing selected rule classes.

System	Informal description	Role
AC	full rule set	baseline dynamics
no_conj	no conjugation rules	conjugation-free dynamics
no_mul	no multiplication rules	reduced length-growth
no_conj_a	partial conjugation restriction	intermediate case
no_conj_b	partial conjugation restriction	intermediate case

Why restrictions are useful

These are mathematically meaningful variants: some have known non-reachability invariants, while others define intermediate dynamics whose behaviour is informative in its own right.

Static QR clouds: first evidence of structure

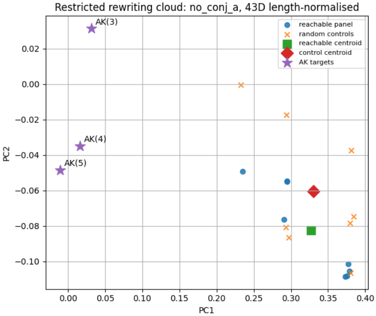


Figure: Length-normalised 43D QR cloud in a restricted rewriting system.

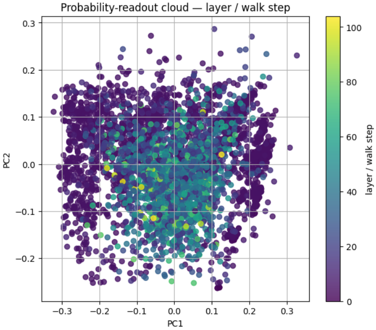


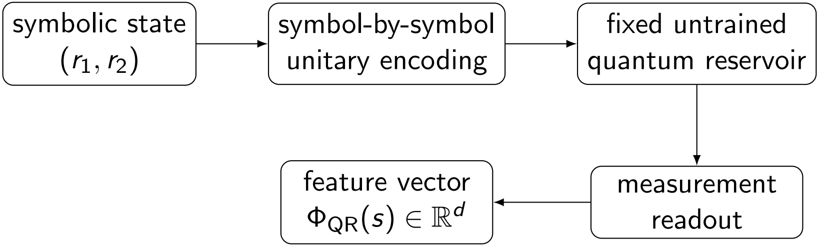
Figure: Probability-readout QR cloud across walk/layer steps.

Message

Static QR embeddings already produce structured feature clouds. The present talk moves from static clouds to temporal structure along rewriting trajectories.

Quantum-reservoir feature map

A quantum reservoir can be viewed as a fixed nonlinear feature map: a symbolic input is encoded into a quantum system, evolved by a fixed circuit, and measured to obtain classical features.

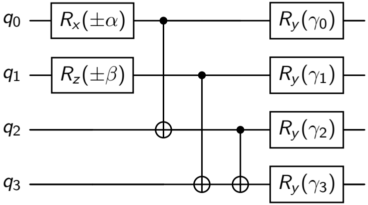


- The reservoir is fixed and untrained.
- Nonlinearity comes from quantum evolution and measurement readouts.
- The QR map is used as a fixed representation layer; clustering, classification, anomaly detection, and time-series methods can then be applied to the resulting trajectories.

QR implementation: one repeated reservoir layer

Encoding idea

A rewriting state $s = (r_1, r_2)$ is encoded symbol-by-symbol into a fixed 4-qubit reservoir. The circuit shows the repeated block applied after one symbol injection.



- a, A are encoded by $R_x(\pm\alpha)$, and b, B by $R_z(\pm\beta)$.
- After each symbol, the fixed entangling and $R_y(\gamma_i)$ gates provide the mixing layer.
- The two relators are processed sequentially; a separator operation is inserted between r_1 and r_2.

QR readouts: from quantum state to feature vector

After processing a rewriting state $s = (r_1, r_2)$, the final 4-qubit state is converted into a classical feature vector

$$\Phi_{\mathrm{QR}}(s) \in \mathbb{R}^d.$$

16D probability readout

Computational-basis probabilities:

$$(p_{0000}, \ldots, p_{1111}).$$

- direct measurement distribution;
- physically interpretable;
- coarse trajectory geometry.

43D derived readout

The 16 probabilities are enriched by derived features:

- entropy and distribution moments;
- single-qubit Z expectations;
- pairwise ZZ expectations;
- covariances / Walsh-type features.

Trajectory structure: shuffled baselines

For a QR feature trajectory

$$x_0, x_1, \ldots, x_T,$$

we define local windows

$$W_i = (x_i, \ldots, x_{i+m-1}).$$

Why use a shuffled baseline?

We want to test whether the *order* of points along a rewriting trajectory carries information, not only whether the unordered cloud of QR features has some structure.

The shuffled baseline is obtained by permuting local trajectory windows.

- It preserves local feature statistics.
- It destroys long-range temporal organisation.
- It gives a comparison point for recurrence and revisitation.

Recurrence ratio

For each window W_i, let

$$D_i = \min_{j \not\approx i} \mathrm{dist}(W_i, W_j)$$

be its nearest nontrivial matching-window distance.

We summarise a trajectory by the mean nearest-window distance

$$S = \frac{1}{N} \sum_i D_i.$$

We then compare the real trajectory with a shuffled baseline:

$$R = \frac{S(\text{real trajectory})}{S(\text{shuffled trajectory})}.$$

Interpretation

$$R < 1 \quad \Longrightarrow \quad \text{stronger recurrence in genuine rewriting trajectories.}$$

Matrix profiles

Matrix profile idea

For each window W_i, record the distance to its closest matching window elsewhere in the trajectory.

$$MP(i) = \min_{j \not\approx i} \mathrm{dist}(W_i, W_j).$$

Low matrix-profile values correspond to:

- repeated motifs;
- revisitation of similar feature-space regions;
- temporal regularities in rewriting dynamics.

Why useful here?

Matrix profiles provide a machine-learning/time-series method for detecting trajectory-level structure in symbolic rewriting dynamics.

Representative results: raw matrix-profile ratios

Rule system	16D QR	43D QR	Length-only
full	0.855	0.461	0.288
no_conj_a	0.907	0.430	0.243
no_conj_b	0.879	0.464	0.237
no_conj	0.924	0.488	0.320
no_mul	0.907	0.607	0.247

Reading the table

Ratios compare real trajectories with shuffled baselines. Smaller values indicate stronger recurrence/revisitation structure.

- 16D QR readout already detects temporal organisation.
- 43D QR readout gives substantially stronger recurrence signals.
- Length evolution is a strong baseline, but QR features retain system-dependent structure beyond length alone.

What the results suggest

- Genuine rewriting trajectories show recurrence beyond shuffled trajectory baselines.
- Different rule systems produce distinct recurrence signatures.
- The 43D observable-derived readout is more sensitive than the 16D probability readout.
- The no-multiplication system exhibits the weakest recurrence signal among the tested systems.
- Multiplication appears to drive large-scale expansion–reduction motifs.
- Conjugation restrictions alter trajectory signatures, consistent with orbit-like revisitation behaviour.

Main message

QR feature trajectories preserve meaningful dynamical information about symbolic rewriting systems.

Machine-learning perspective

The framework combines:

Component	Role
QR feature map	nonlinear representation of symbolic states
Trajectory analysis	dynamics in continuous feature space
Matrix profiles	motif and recurrence detection
Controls/baselines	distinguish genuine temporal structure

Potential ML directions:

- clustering of rewriting regimes;
- nearest-neighbour classification;
- anomaly detection;
- trajectory motif discovery;
- ML-assisted discovery of candidate dynamical invariants.

Limitations and interpretation

What this does not yet provide

- no formal proof of non-reachability;
- no extracted invariant yet;
- no claim of quantum speed-up.

What it does provide

- a nonlinear feature-space view of symbolic rewriting trajectories;
- measurable recurrence signatures;
- empirical distinction between rule systems;
- a possible route toward ML-assisted invariant discovery.

Exploratory, but structurally informative.

Outlook

Natural next steps:

- larger and more systematic trajectory datasets and longer rewriting runs;
- comparison with classical random feature maps;
- dynamic or history-dependent quantum reservoirs;
- graph/tree matrix profiles for branching rewriting systems;
- supervised and unsupervised learning on trajectory signatures;
- symbolic interpretation of discovered feature-level regularities.

Longer-term aim

Use ML-assisted QR trajectory analysis to suggest candidate geometric or dynamical invariants for symbolic reachability and non-reachability.

Summary

- We embed symbolic rewriting states into fixed quantum-reservoir feature spaces.
- Rewriting paths become continuous feature trajectories.
- Recurrence ratios and matrix profiles detect temporal organisation.
- Different Andrews–Curtis rule systems have distinct trajectory signatures.
- The approach suggests a possible ML-assisted route toward invariant discovery.

Thank you!