

A Computational Framework for Automated Discovery within Conservative Matrix Fields (CMF)

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Monderer, Elyasheev Leibtag, Rotem Kalisch, Ido Kaminer

Mathematical constants are everywhere!

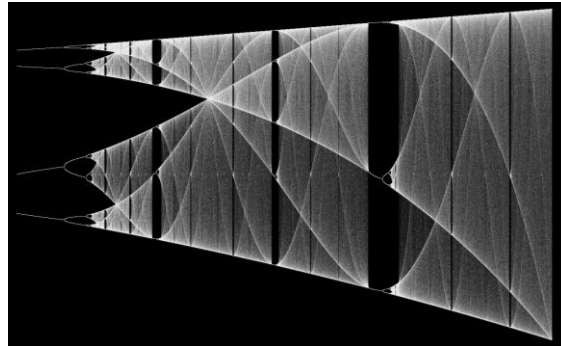
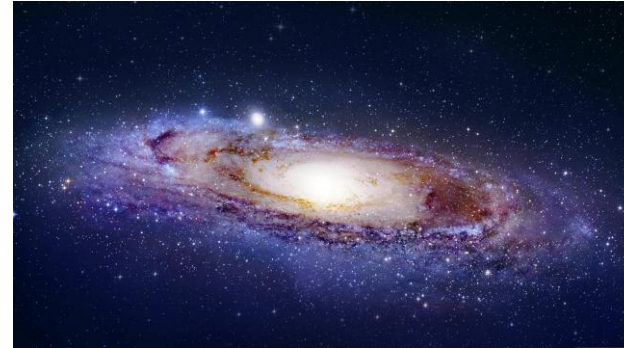
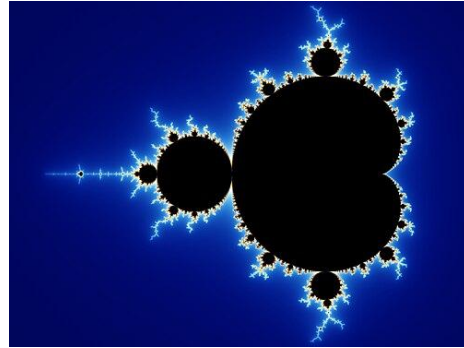
$$\pi = \lim_{\epsilon \rightarrow 0} (N \cdot \sqrt{\epsilon})$$

$$K_0 = \prod_{n=1}^{\infty} \left(1 + \frac{1}{n(n+2)} \right)^{\frac{\ln n}{\ln 2}}$$

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} \quad \zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}$$

$$\delta = \lim_{n \rightarrow \infty} \frac{a_{n-1} - a_{n-2}}{a_n - a_{n-1}}$$

$$M = \sum_{i,j,k=-\infty}^{\infty} \frac{(-1)^{i+j+k+1}}{\sqrt{i^2 + j^2 + k^2}}$$



$$\mu = \lim_{n \rightarrow \infty} \sqrt[n]{c_n}$$

$$\gamma = \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n \frac{1}{k} - \ln n \right)$$

$$G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2}$$

$$\varphi = \frac{1 + \sqrt{5}}{2}$$

$$\delta = \int_0^{\infty} \frac{e^{-x}}{1+x} dx$$

But finding a formula or proving
irrationality is not that easy...



Ramanujan

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{k=0}^{\infty} \frac{(4k)! (1103 + 26390k)}{(k!)^4 396^{4k}}$$



Gauss

$$\pi = \lim_{n \rightarrow \infty} \frac{4}{1 + \frac{1^2}{3 + \frac{2^2}{5 + \frac{3^2}{\ddots + \frac{n^2}{2n+1}}}}}$$



Euler

$$e = \frac{1}{1 - \frac{1}{2 - \frac{1}{3 - \frac{2}{4 - \frac{3}{5 - \dots}}}}}$$



Leibniz

$$\frac{\pi}{4} = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}$$

But finding a formula or proving
irrationality is not that easy...

$$\frac{4}{\pi - 2} = 3 + \frac{3}{5 + \frac{8}{7 + \frac{15}{9 + \frac{24}{11 + \dots}}}}$$



Ramanujan

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{k=0}^{\infty} \frac{(4k)! (1103 + 26390k)}{(k!)^4 396^{4k}}$$

$$\frac{1}{2G-1} = \frac{1}{2} + \frac{\frac{1 \cdot 1}{2} + \frac{1 \cdot 2}{2^2}}{\frac{1}{2} + \frac{1}{\frac{2 \cdot 3}{2} + \dots}}$$



Euler

$$e = \frac{1}{1 - \frac{1}{2 - \frac{1}{3 - \frac{2}{4 - \frac{3}{5 - \dots}}}}}$$

$$\pi = \frac{4}{1 + \frac{1^2}{2 + \frac{3^2}{2 + \frac{5^2}{\dots}}}}$$

$$\frac{\pi}{2} = \prod_{n=1}^{\infty} \frac{(2n)^2}{(2n-1)(2n+1)}$$



Gauss

$$\frac{1}{2-2G} = 3 + \frac{2^2}{1 + \frac{2^2}{3 + \frac{4^2}{1 + \frac{4^2}{\dots}}}}$$

$$\pi = \lim_{n \rightarrow \infty} \frac{4}{1 + \frac{1^2}{3 + \frac{2^2}{5 + \frac{3^2}{\ddots + \frac{n^2}{2n+1}}}}}$$



Leibniz

$$\frac{\pi}{4} = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}$$

$$\pi = \sqrt{12} \sum_{k=0}^{\infty} \frac{(-3)^{-k}}{2k+1}$$

$$\pi = 2 + \frac{2}{1 + \frac{1 \cdot 2}{1 + \frac{2 \cdot 3}{1 + \frac{3 \cdot 4}{1 + \dots}}}}$$

$$\frac{5}{4} - \zeta(3) = \sum_{n=2}^{\infty} \frac{1}{n^3(n^2-1)}$$

Our Goal – I

Fast Convergence



Srinivasa
Ramanujan

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{k=0}^{\infty} \frac{(4k)! (1103 + 26390k)}{(k!)^4 396^{4k}}$$

- M. Shalyt et al. Unifying Structure for Ramanujan's 17 Series for $1/\pi$: A Human-AI Discovery. *RISC Proceedings on Symbolic Computation and Machine Learning*, 3:82-83, 2026.
- R. Hemmecke et al. Computer-Assisted Construction of Ramanujan-Sato Series for 1 over π . *RISC Report Series 25-01*, 2025.
- S. Ramanujan. Modular equations and approximations to π . *Quarterly Journal of Mathematics*, 45:350-372, 1914.

Our Goal – II

Irrationality



Roger Apéry

$$(n + 1)^3 x_{n+1} = (34n^3 + 51n^2 + 27n + 5)x_n - n^3 x_{n-1}$$

Take:

$$A_0 = 1, A_1 = 5 \quad \text{and} \quad B_0 = 0, B_1 = 6$$

Then:

$$B_n - \zeta(3)A_n \sim (\sqrt{2} - 1)^{4n}$$

Hence:

$$\frac{A_n}{B_n} \rightarrow \zeta(3)$$

R. Apéry. Irrationalité de $\zeta(2)$ et $\zeta(3)$. *Astérisque*, 61:11–13, 1979.

Approximations & Irrationality

Define Irrationality measure of an approximation:

$$\delta = \lim_{n \rightarrow \infty} \delta_n, \quad \delta_n = -1 - \frac{\log \left| x - \frac{p_n}{q_n} \right|}{\log |q_n|}$$

In Apéry's case:

$$p_n = \frac{A_n}{d_n}, \quad q_n = \frac{B_n}{d_n}, \quad d_n = \text{lcm}(1, \dots, n)^{-3} \sim e^{-3n}$$

Then:

$$\left| \frac{p_n}{q_n} - \zeta(3) \right| < \frac{1}{|q_n|^{1+\delta}} \text{ where } \delta = 0.0805 > 0 \Rightarrow \zeta(3) \notin \mathbb{Q}$$

$\Rightarrow$ We want to optimize for δ !

R. Apéry. Irrationalité de $\zeta(2)$ et $\zeta(3)$. *Astérisque*, 61:11–13, 1979.

Linear Recurrences

$$u_n = a_{1,n}u_{n-1} + a_{2,n}u_{n-2} + \cdots + a_{m,n}u_{n-m}$$

General form:

$$u_n = a_n u_{n-1} + b_n u_{n-2}$$

a_n	b_n	Limit
$n^3 + (n+1)^3$	$-n^6$	$\zeta(3)^{-1}$
$n^3 + (n+1)^3 + 4(2n+1)$	$-n^6$	$(\zeta(3) - 1)^{-1}$
$3n^2 + 3n + 1$	$-2n^4$	$(2G)^{-1}$
$n^3 + (n+1)^3 + (n+1)^2$	$n^5(n+1)$	$(\zeta(3) - \zeta(2) + 1)^{-1}$

Take solutions: $p_{-1} = 1, p_0 = a_0, q_{-1} = 0, q_0 = 1$ then $\frac{p_n}{q_n} \rightarrow L$

R. Elimelech et al. Algorithm-assisted discovery of an intrinsic order among mathematical constants. *Proceedings of the National Academy of Sciences*, 121(25):e2321440121, 2024.

What was found

Unify formulas for π
+
Connect D-finite functions

$$\pi = \sum_{k=0}^{\infty} \left[\frac{1}{16^k} \left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6} \right) \right]$$

Bailey-Borwein-Plouffe

Mathematics of Computation (1997)

$$\pi = \frac{4}{1 + \frac{1^2}{3 + \frac{2^2}{5 + \frac{3^2}{7 + \ddots}}}}$$

Carl Friedrich Gauss
1813, Germany

$$\frac{\pi}{2} = \prod_{n=1}^{\infty} \frac{(2n)^2}{(2n-1)(2n+1)}$$

John Wallis
1656, England.

π

$$\pi = \sqrt{12} \sum_{k=0}^{\infty} \frac{(-3)^{-k}}{2k+1}$$

Madhava of Sangamagrama
14th century, India.

$$\frac{4}{\pi-2} = 3 + \frac{3}{5 + \frac{8}{7 + \frac{15}{9 + \frac{24}{11 + \ddots}}}}$$

Raayoni et al.
Nature (2021)

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{k=0}^{\infty} \frac{(4k)!(1103 + 26390k)}{(k!)^4 396^{4k}}$$

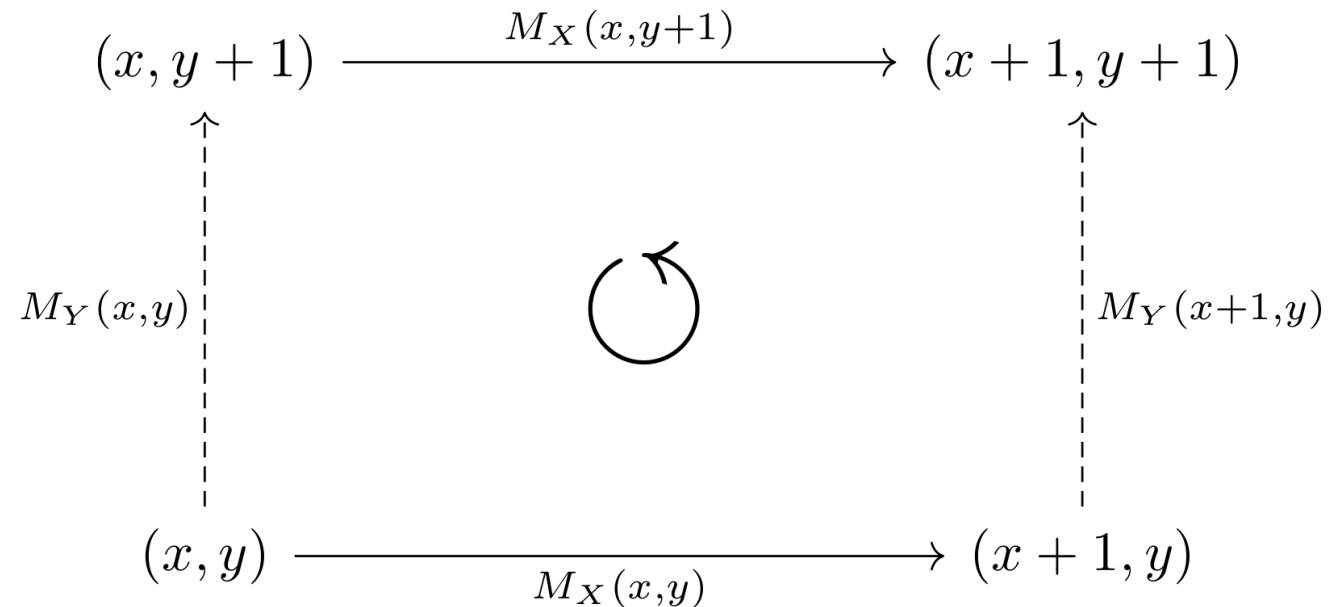
Srinivasa Ramanujan
1914, India.

T. Raz et al. From Euler to AI: Unifying formulas for mathematical constants. In *Advances in Neural Information Processing Systems*, 2025.

S. Weinbaum et al. On conservative matrix fields: Continuous asymptotics and arithmetic, 2025.

What is a CMF?

$$M_Y(x, y)M_X(x, y + 1) = M_X(x, y)M_Y(x + 1, y)$$



R. Elimelech et al. Algorithm-assisted discovery of an intrinsic order among mathematical constants. *Proceedings of the National Academy of Sciences*, 121(25):e2321440121, 2024.

What is a CMF?

Example – CMF from a Generalized Hypergeometric:

$${}_pF_q \left[\begin{matrix} x_1 & \dots & x_p \\ x_{p+1} & \dots & x_{p+q} \end{matrix}; z \right] = \sum_{n=0}^{\infty} \frac{(x_1)_n \dots (x_p)_n}{(x_{p+1})_n \dots (x_{p+q})_n} \frac{z^n}{n!}$$

From ODE and contiguous relations $\rightarrow M_{\theta_z}^{pFq}$ (Companion Matrix):

$$\forall 1 \leq i \leq p: M_{e_i} = \frac{1}{x_i} M_{\theta_z}^{pFq} + I,$$

$$\forall p+1 \leq i \leq p+q: M_{-e_j} = \frac{1}{x_i - 1} M_{\theta_z}^{pFq} + I$$

$$(a)_n := \prod_{k=0}^{n-1} (a+k)$$

S. Weinbaum et al. On conservative matrix fields: Continuous asymptotics and arithmetic, 2025.

CMF Limit

So how do we get our $\frac{p_n}{q_n}$ actually?

Define CMF limit as:

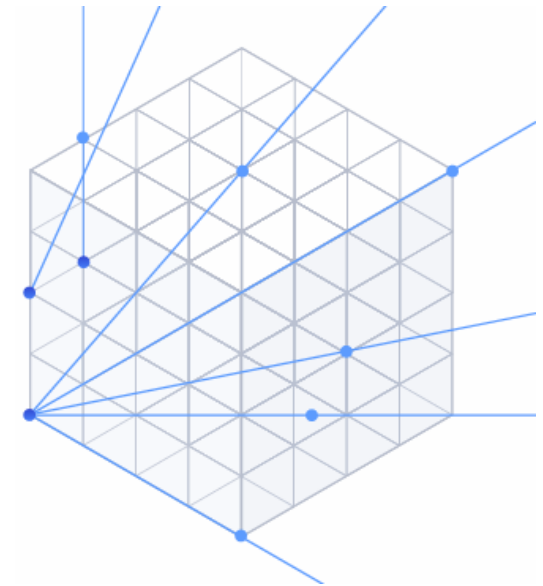
$$\mathcal{L}_{x,v}^{p',q'}(n) = \frac{p'^T \cdot \mathcal{M}_{nv}(x) \cdot e_r}{q'^T \cdot \mathcal{M}_{nv}(x) \cdot e_r} = \frac{p_n}{q_n}$$

Where:

- $v :=$ trajectory
- $x :=$ start point
- $\mathcal{M}_{nv}(x) :=$ “walk” matrix starting from x and walking up to $x + nv$.

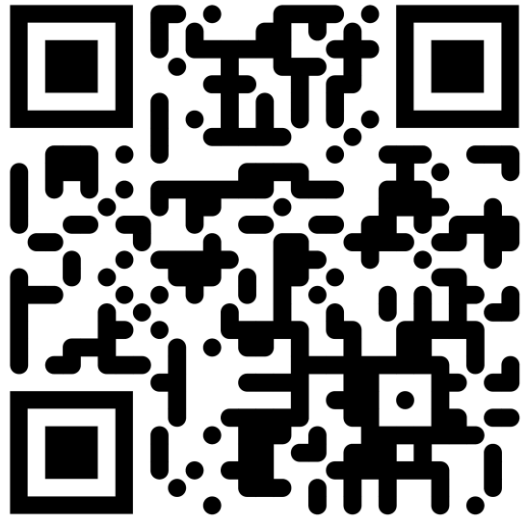
And how do we get p', q' ? - PSLQ

S. Weinbaum et al. On conservative matrix fields: Continuous asymptotics and arithmetic, 2025.



And Practically?

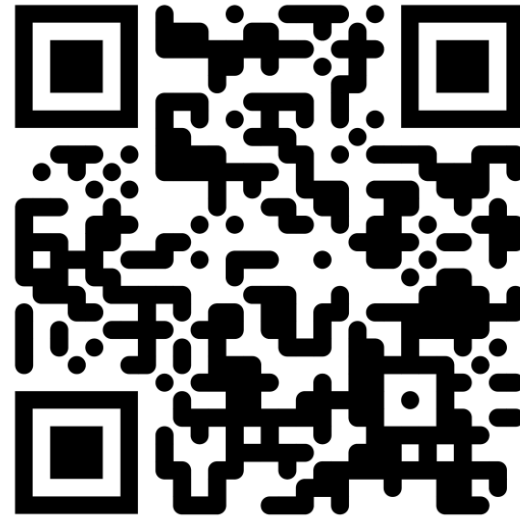
Create and interact with CMFs



Ramanujan Tools
By Rotem Kalisch
(and others)



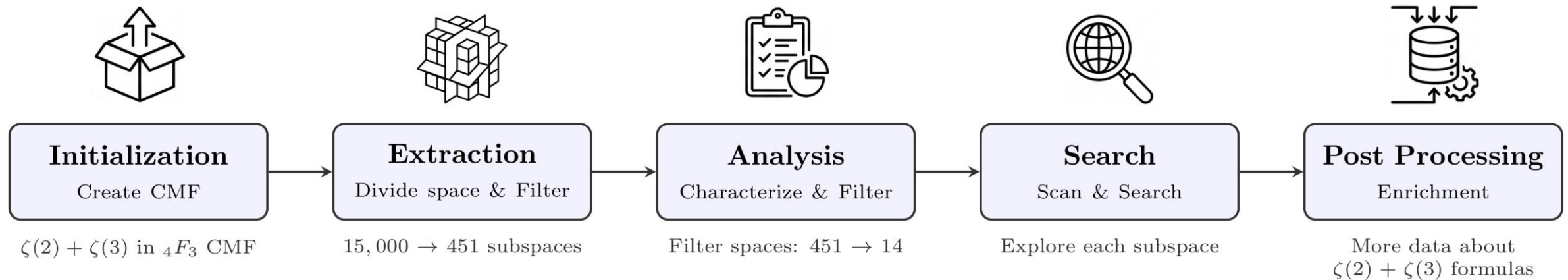
Search and explore new formulas



Ramanujan's Dreams
By Me!



Our Solution - Overview



Reminder: What do we want to achieve?

- Search for hints of irrationality
- Recurrence relation data mining



Creation

Example: ${}_2F_1(z = -1)$ searching for $\log(2)$.

$$CM := \begin{pmatrix} 0 & -\frac{xy}{2} \\ 1 & -\frac{x+y+z-1}{2} \end{pmatrix}$$

Hence:

$$M_x := \begin{pmatrix} 0 & -\frac{y}{2} \\ \frac{1}{x} & \frac{x-y-z+1}{2x} \end{pmatrix}, \quad M_y := \begin{pmatrix} 0 & -\frac{x}{2} \\ \frac{1}{y} & \frac{-x+y-z+1}{2y} \end{pmatrix}, \quad M_z := \frac{z}{(x-z)(y-z)} \begin{pmatrix} -x-y+z & xy \\ -2 & 2z \end{pmatrix}$$



Creation

$$M_x := \begin{pmatrix} 0 & -\frac{y}{2} \\ \frac{1}{x} & \frac{x-y-z+1}{2x} \end{pmatrix}$$

$$M_y := \begin{pmatrix} 0 & -\frac{x}{2} \\ \frac{1}{y} & \frac{-x+y-z+1}{2y} \end{pmatrix}$$

$$M_z := \frac{z}{(x-z)(y-z)} \begin{pmatrix} -x-y+z & xy \\ -2 & 2z \end{pmatrix}$$

Hypergeometric

An Infinite Family of Apéry-like Recurrences of Euler's Constant

An Infinite Family of Apéry-like Recurrences for Euler's Constant
 Rotem Kalisch, Tali Monderer, Elyashev Leibtag, Shachar Weinbaum, Michael Shalyt, Ido Kaniner
 Technion - Israel Institute of Technology, Haifa 3200003, Israel

Motivation
 Euler's constant $\gamma = 0.57721 \dots$ remains one of the most elusive constants in mathematics. Its irrationality is still unknown and very few Apéry-like recurrences are known to approximate it [1, 2, 3, 4].
Goal: generate infinitely many Apéry-like recurrences converging to γ .

The Meijer G-function CMF
 A Conservative Matrix Field (CMF) [5] is a collection of basis matrices $M_i(x): \mathbb{R}^+ \rightarrow GL_n(\mathbb{Q})$, $i = 1, \dots, d$, satisfying the commutation relation $M_i(x)M_j(x) = M_j(x)M_i(x)$.

A Meijer G Identity
 Let $\gamma \in \mathbb{N}$, $\theta = \frac{\gamma}{2}$ and define $G_\theta(z) = G_{\theta, \theta}^{0, 0} \left[\begin{matrix} 0, \dots, 0 \\ 0, 0, \dots, 0 \end{matrix} \middle| z \right]$. Define the two differential operators $N_\theta = (1 + \theta^2 z^2 + (-1)^{\theta-1} z^{\theta-1})$, $D_\theta = (1 + \theta^2 z^2 + (-1)^{\theta-1} z^{\theta-1})$. Then $N_\theta[G_\theta](1) = \gamma$, $D_\theta[G_\theta](1) = 1$.

An Example of Order 3
 Let $\gamma_0 = (0, 0, 1, 1, 1)$, yielding $M_{\gamma_0}(n) = \begin{bmatrix} 1 & n^2 & (n-1)(3n+3)(n^2+n+1) \\ 2 & 1-3n^2 & -(n+1)(3n^2+n+5) \\ 1 & 3n+2 & 6n(n+1) \end{bmatrix}$. For a generic seed, for instance $\gamma^* = (0, 0, 1)$, the dual iteration satisfies $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} T_{\gamma^*} \approx \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ($n \rightarrow \infty$).

Numerical Behavior
 We compare trajectories by the error $\| \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} M_{\gamma}(n) T_{\gamma} - \gamma \|$. Trajectories in the proven family $\mathcal{T}$ are marked by an asterisk. The recurrence by Apolukarev et al. [1] is shown as a reference point.

Main Result
 We construct a Meijer G CMF of rank r and lattice dimension $d = 3r - 1$. For trajectories γ in the proven infinite family, and conjecturally in the broader higher-rank family, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} M_{\gamma}(n) T_{\gamma} \rightarrow \gamma$ as $n \rightarrow \infty$.

Dual Convergence
 If $\Delta(\gamma)$ is the unique recursive solution of the forward system, then by Miller's algorithm [6] the dual system isolates the initial state vector.

References
 [1] Alexander I. Apolukarev, "On linear forms containing the Euler constant", In: *arXiv preprint arXiv:0902.1748* (2009).
 [2] Torgny Riebel, "Rational approximations for values of derivatives of the Gamma function", In: *Transactions of the American Mathematical Society* 361.12 (2009), pp. 4133–4149.
 [3] Kh. Hissam Pilehrood and T. Hissam Pilehrood, "On a continued fraction expansion for Euler's constant", In: *Journal of Number Theory* 133.2 (2013), pp. 748–756.
 [4] Jeffrey Lagarias, "Euler's constant: Euler's work and modern developments", In: *Bulletin of the American Mathematical Society* 54 (2017), pp. 327–328.
 [5] Shachar Weinbaum et al., "On conservative matrix fields: Continuous asymptotics and arithmetic", In: *arXiv preprint arXiv:2307.08121* (2023).
 [6] Ray VM Zahar, "A mathematical analysis of Miller's algorithm", In: *Numerische Mathematik* 27.4 (1976), pp. 427–447.

Symbolic Computation and Differential and Difference Equations, Hagenberg, Austria, 8 - 10 July, 2024

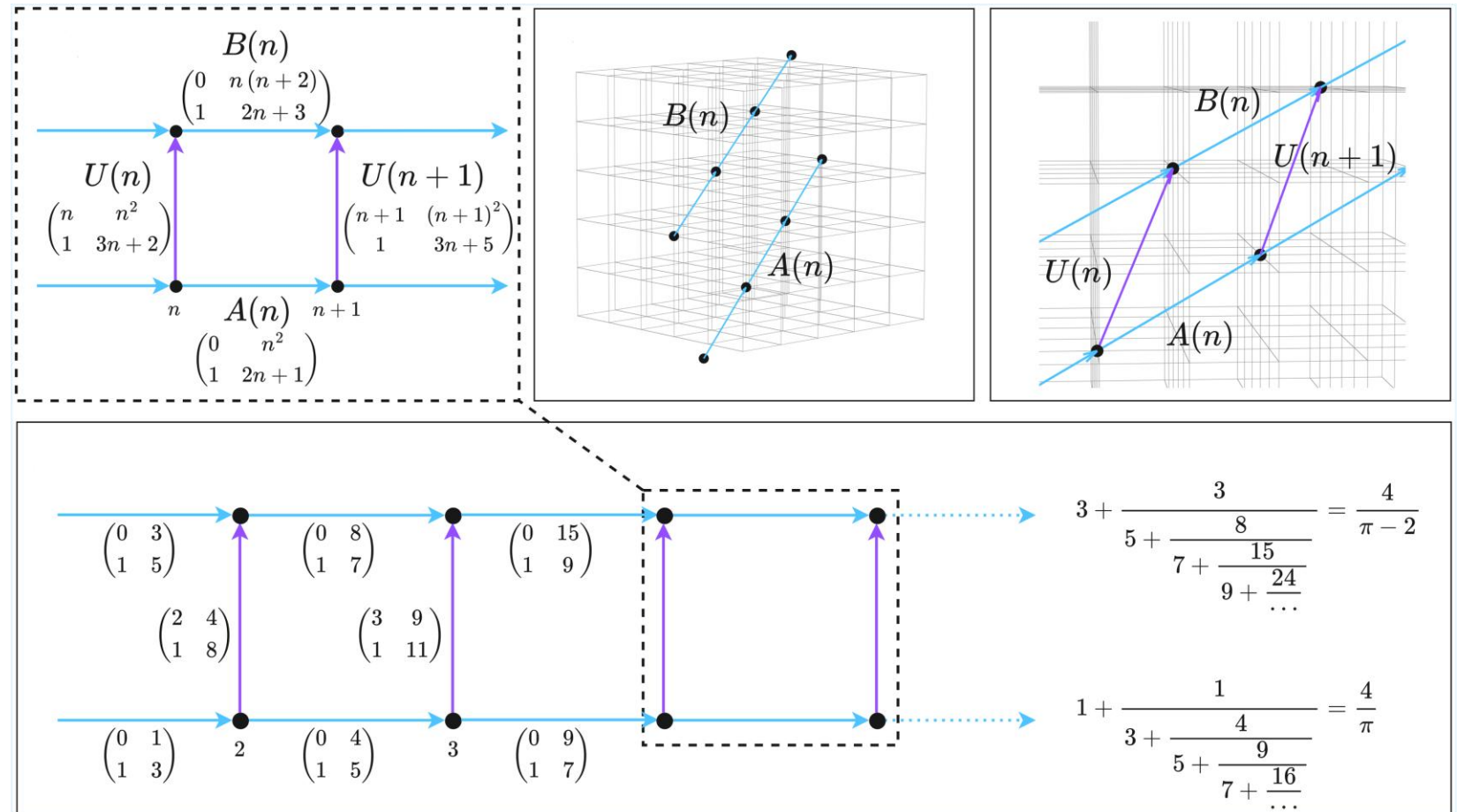
Rotem Kalisch

Meijer G

Extraction

Require

1. Want to use full space – step in negative directions.
2. Use a single start point.





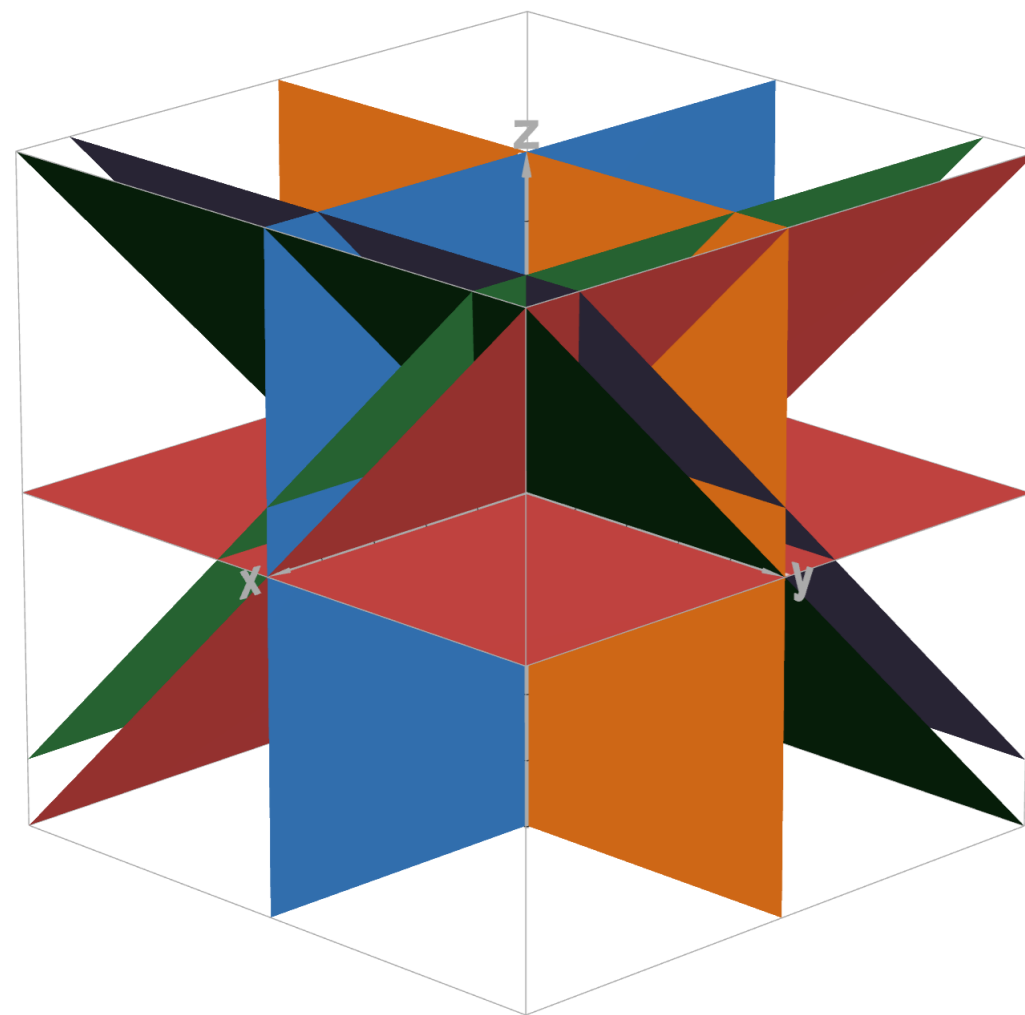
Extraction

CMF “walls”:

$$\begin{array}{lll} x = 0, & x - z = 0, & x - z = -1 \\ y = 0, & y - z = 0, & y - z = -1 \\ & z = 0 & \\ & \downarrow & \\ & Ax = b & \end{array}$$

Shard: Unbounded chamber $A'x < b$ defined by the hyperplanes $:= \{M_i\text{'s poles} \cup \det(M_i)\text{'s zeros}\}$.

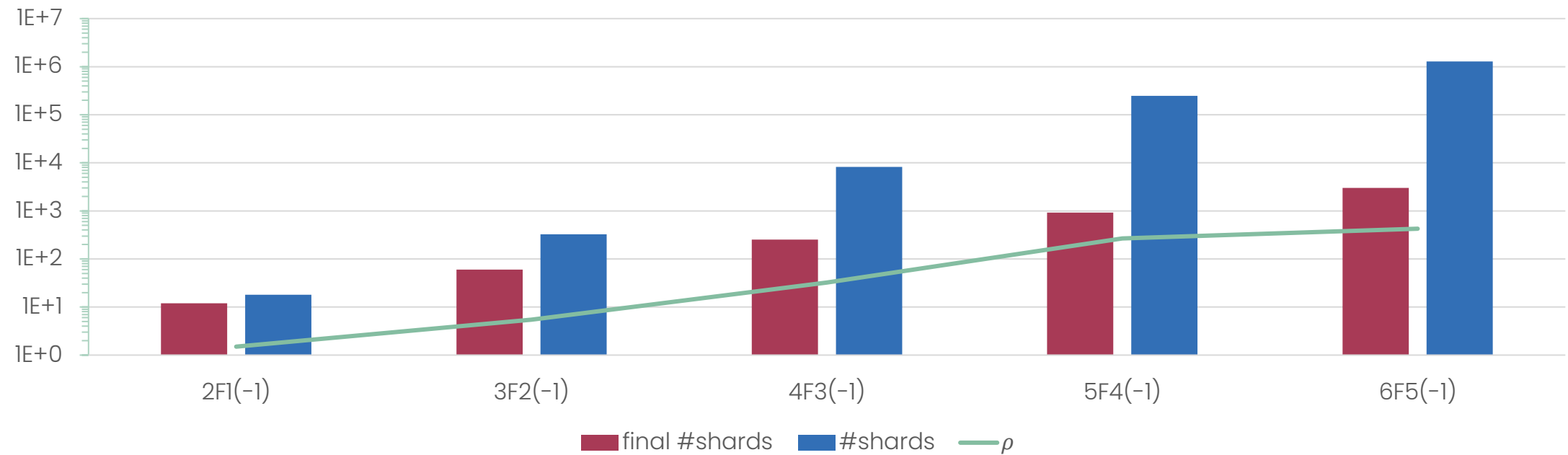
Goal: find all shards in the CMF





Symmetry Shard Reduction

Effectiveness of Symmetry Application on Generalized Hypergeometric CMFs



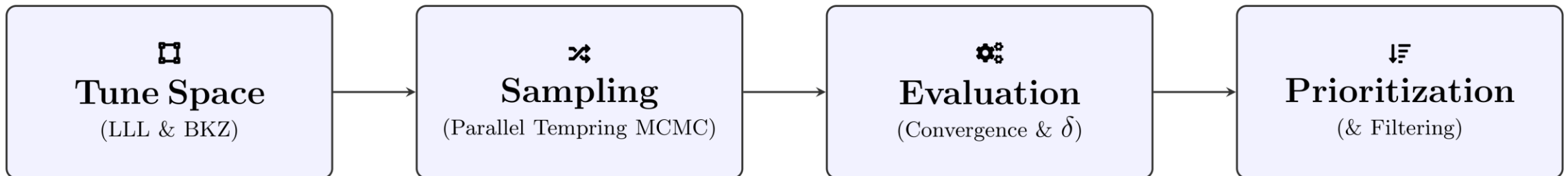
$$\rho = \frac{\#shards}{\text{final } \#shards}$$



Analysis

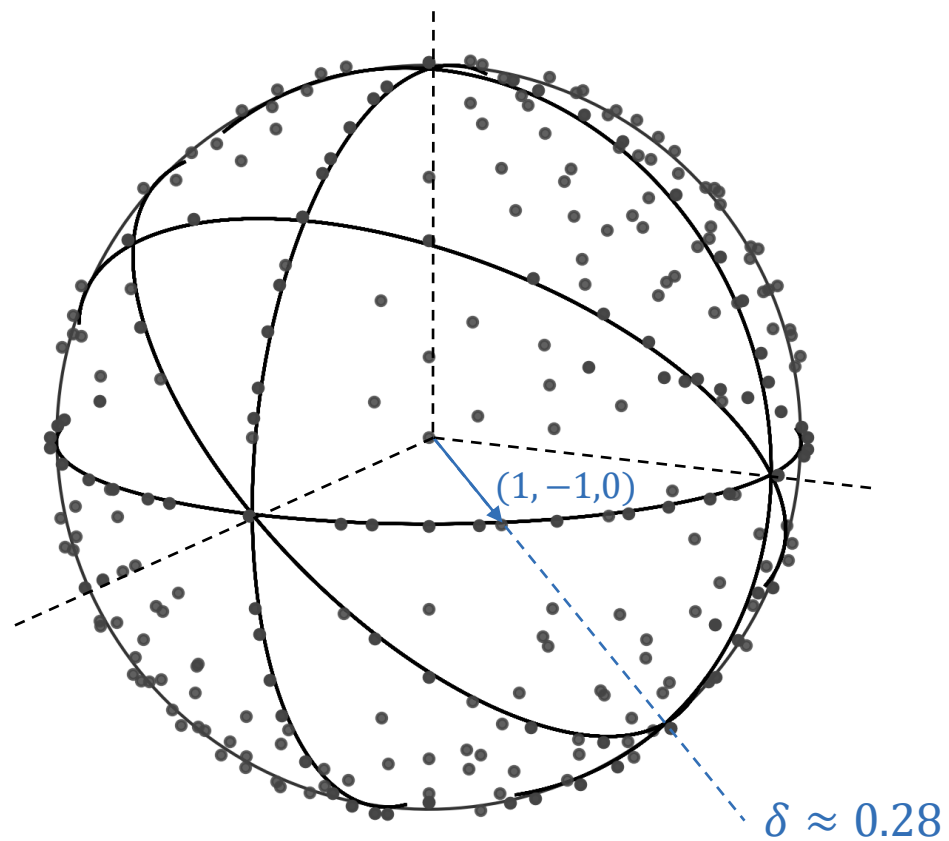
- Want trajectory v s.t. v is a recession direction.
i.e.: v stays in a shard defined by (A, b) if $Av \leq 0$
- Which Shards are interesting?

Check convergence to constant and other attributes $\rightarrow$ sample and scan!



Analysis

$$-(n+2)(2n+5)u_n = 12(n+3)^2u_{n-1} - (n+4)(2n+7)u_{n-2}$$





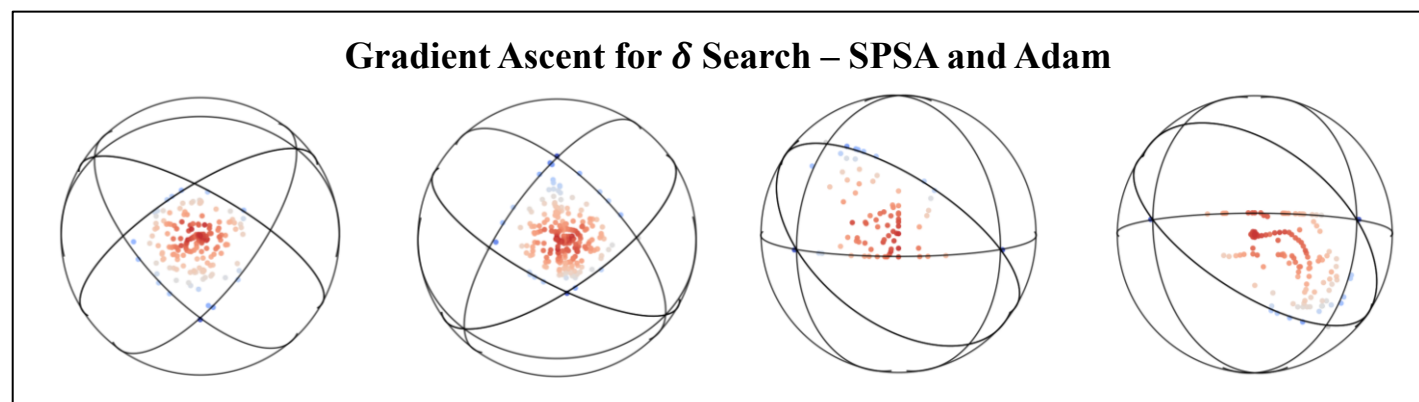
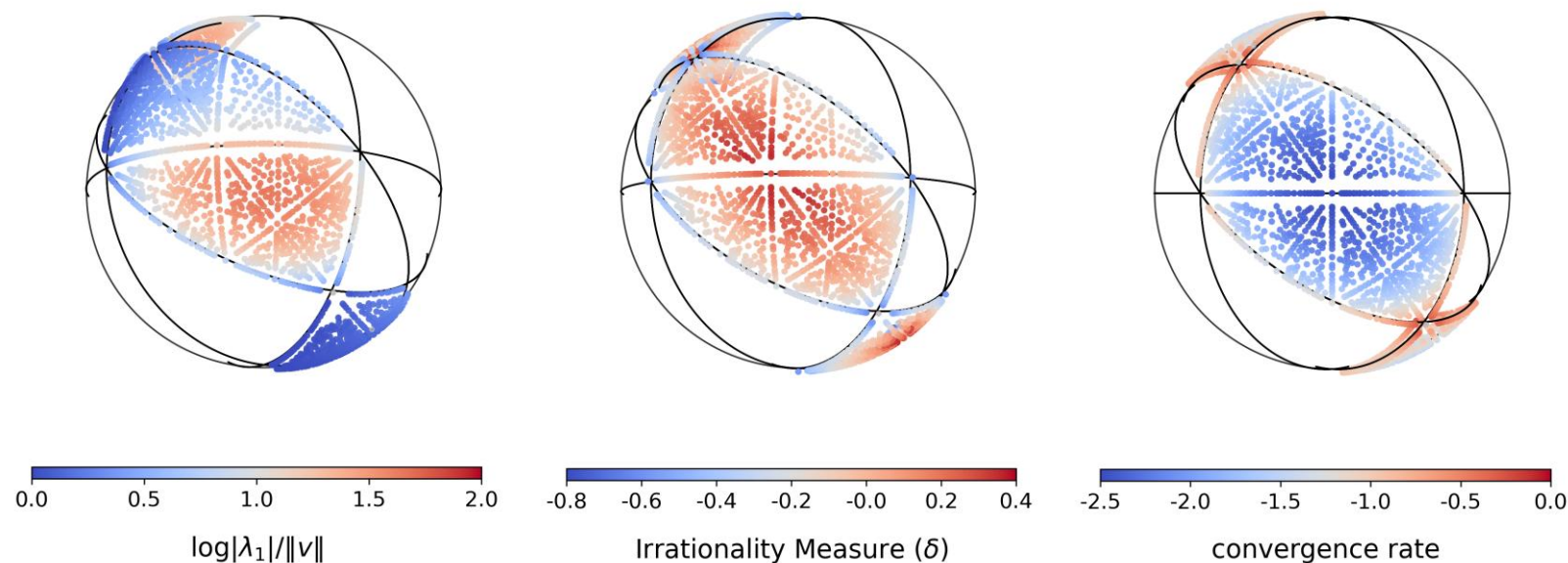
Search

Methods:

- Broad scan
- Simulated Annealing
- Genetic algorithm

Utilizing continuity:

- Gradient Ascent

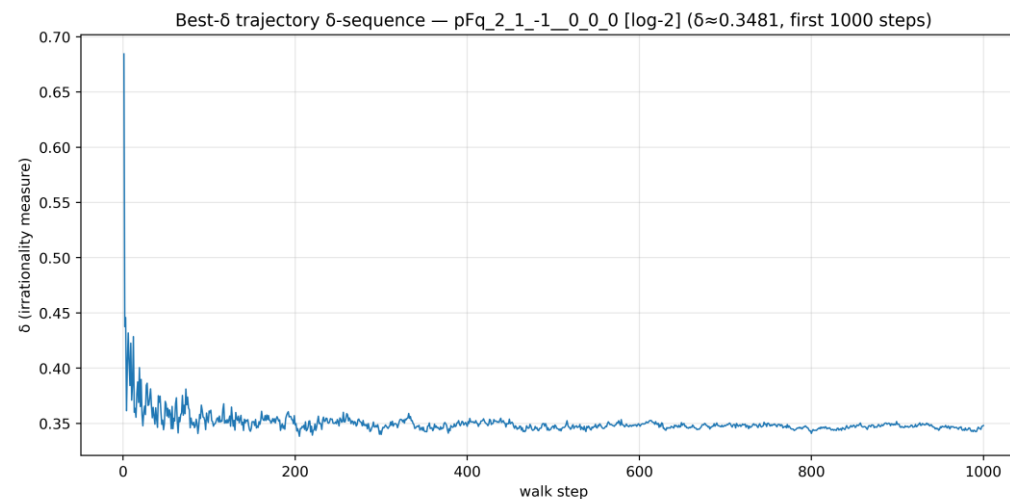
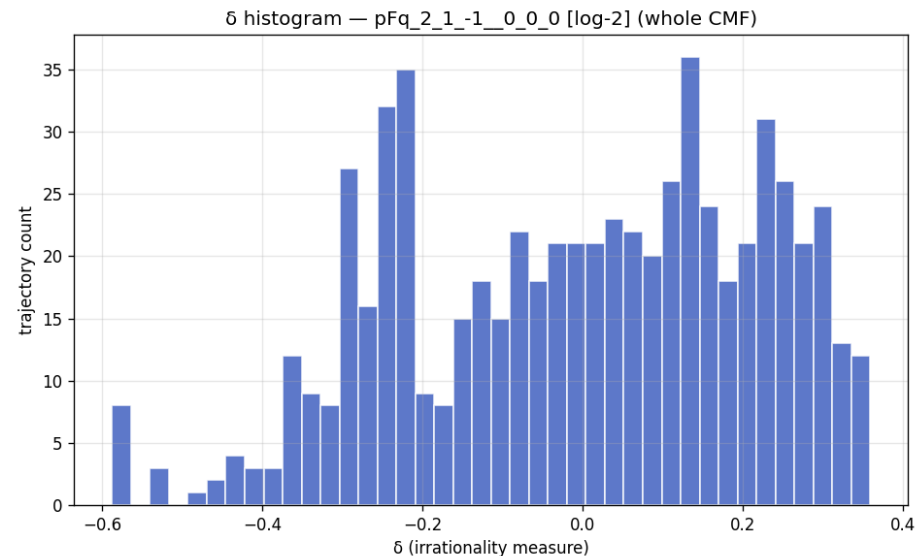


Continuity conjectures: S. Weinbaum et al. On conservative matrix fields: Continuous asymptotics and arithmetic, 2025.



Post Processing

- Attributes:
 - Symbolic formula representation
 - Digits per step
 - $\gcd(p_n, q_n)$ slope
 - Convergence rate
 - Eigen values
 - ...
- Graphs (statistics)

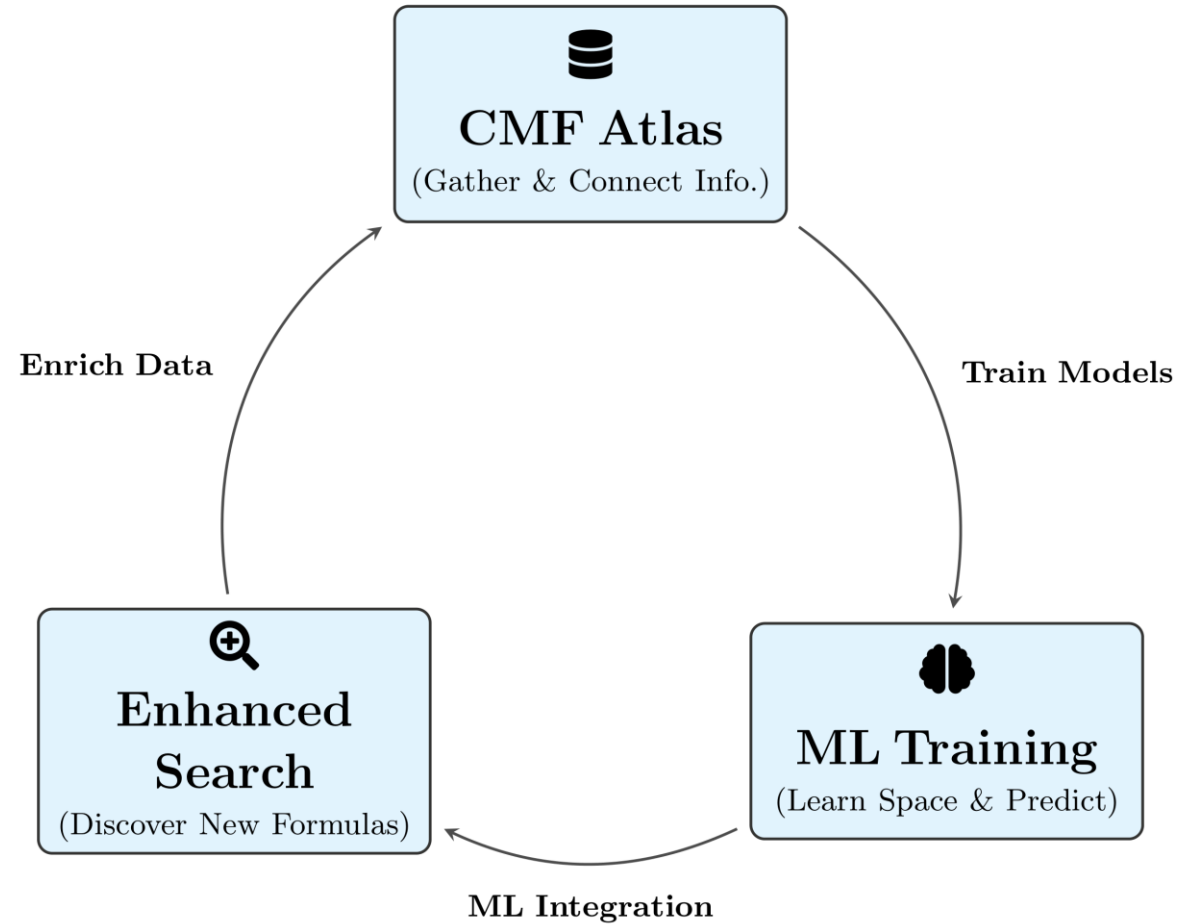


Findings

Constant	CMF	δ	Dimensions	
G	${}_4F_3(\mathbf{a}, \mathbf{b}; 1)$	-0.275	7	WR δ improvement
$\zeta(2)$	${}_4F_3(\mathbf{a}, \mathbf{b}; 1)$	0.26	7	
$\zeta(2) + \zeta(3)$	${}_4F_3(\mathbf{a}, \mathbf{b}; 1)$	$-0.0002 (\pm 0.001)$	7	
$\frac{4}{3\sqrt{3}}\mathcal{G}_{Gi}$	${}_3F_2(\mathbf{a}, \mathbf{b}; \frac{1}{4})$	-0.06	5	
π	integral	0.16	3	WR δ reconstruction
$\log(2)$	${}_2F_1(\mathbf{a}, \mathbf{b}; -1)$	0.346	3	
$\log(2)$	${}_3F_2(\mathbf{a}, \mathbf{b}; \frac{1}{2})$	0.388	5	
G	${}_3F_2(\mathbf{a}, \mathbf{b}; 1)$	-0.43	5	

Checkout: Hila Barkan – “New Directions in Irrationality”

Vision and Future



Thanks 😊

More details at

- *Weinbaum et al.*, On Conservative Matrix Fields: Continuous Asymptotics and Arithmetic. 2025.
- *Raz et al.*, From Euler to AI: Unifying Formulas for Mathematical Constants. *In Advances in Neural Information Processing Systems*, 2025.
- *Elimelech et al.*, Algorithm-assisted discovery of an intrinsic order among mathematical constants. *Proceedings of the National Academy of Sciences*, 2024.

