A Computational Framework for Automated Discovery within Conservative Matrix Fields (CMF)

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Mathematical constants are everywhere!

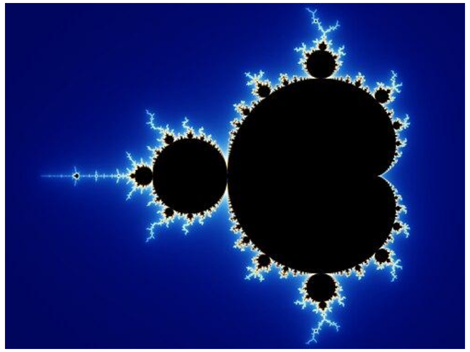
$$\pi = \lim_{\epsilon \to 0}(N \cdot \sqrt{\epsilon})$$

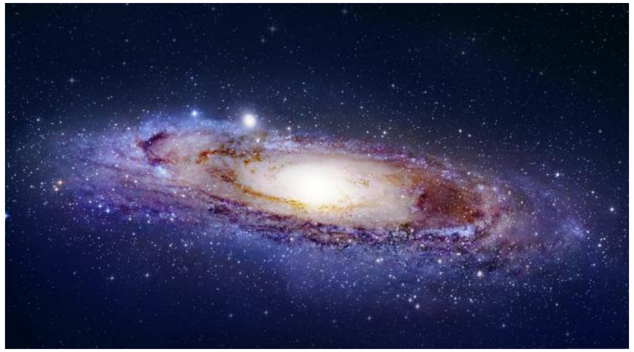
$$K_0 = \prod_{n=1}^{\infty}\left(1 + \frac{1}{n(n+2)}\right)^{\frac{\ln n}{\ln 2}}$$

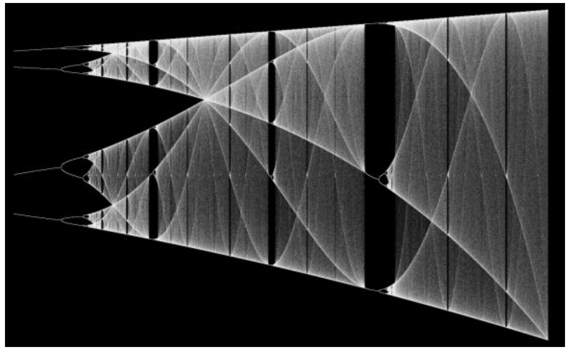
$$e = \sum_{n=0}^{\infty}\frac{1}{n!} \qquad \zeta(3) = \sum_{n=1}^{\infty}\frac{1}{n^3}$$

$$\delta = \lim_{n\to\infty}\frac{a_{n-1} - a_{n-2}}{a_n - a_{n-1}}$$

$$M = \sum_{i,j,k=-\infty}^{\infty}\frac{(-1)^{i+j+k+1}}{\sqrt{i^2 + j^2 + k^2}}$$







$$\mu = \lim_{n\to\infty}\sqrt[n]{c_n}$$

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \ln n\right)$$

$$G = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}$$

$$\varphi = \frac{1+\sqrt{5}}{2}$$

$$\delta = \int_0^{\infty}\frac{e^{-x}}{1+x}dx$$

But finding a formula or proving irrationality is not that easy…

Ramanujan

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801}\sum_{k=0}^{\infty}\frac{(4k)!\,(1103+26390k)}{(k!)^4 396^{4k}}$$

Gauss

$$\pi = \lim_{n\to\infty}\cfrac{4}{1+\cfrac{1^2}{3+\cfrac{2^2}{5+\cfrac{3^2}{\ddots+\cfrac{n^2}{2n+1}}}}}$$

Euler

$$e = \cfrac{1}{1-\cfrac{1}{2-\cfrac{1}{3-\cfrac{2}{4-\cfrac{3}{5-\cdots}}}}}$$

Leibniz

$$\frac{\pi}{4} = \sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}$$

But finding a formula or proving irrationality is not that easy…

$$\frac{4}{\pi-2} = 3 + \cfrac{3}{5+\cfrac{8}{7+\cfrac{15}{9+\cfrac{24}{11+\cdots}}}}$$

$$\frac{1}{2-2G} = 3+\cfrac{2^2}{1+\cfrac{2^2}{3+\cfrac{4^2}{1+\cfrac{4^2}{\cdots}}}}$$

Ramanujan

$$\frac{1}{\pi}=\frac{2\sqrt{2}}{9801}\sum_{k=0}^{\infty} \frac{(4k)!\,(1103+26390k)}{(k!)^4\, 396^{4k}}$$

Gauss

$$\pi = \lim_{n\to\infty} \cfrac{4}{1+\cfrac{1^2}{3+\cfrac{2^2}{5+\cfrac{3^2}{\ddots+\cfrac{n^2}{2n+1}}}}}$$

$$\frac{1}{2G-1} = \frac{1}{2} + \cfrac{1\cdot 1}{\frac{1}{2}+\cfrac{1\cdot 2}{\frac{1}{2}+\cfrac{2^2}{\frac{1}{2}+\cfrac{2\cdot 3}{\cdots}}}}$$

Euler

$$e = \cfrac{1}{1-\cfrac{1}{2-\cfrac{1}{3-\cfrac{2}{4-\cfrac{3}{5-\cdots}}}}}$$

Leibniz

$$\frac{\pi}{4}=\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}$$

$$\pi=\sqrt{12}\sum_{k=0}^{\infty} \frac{(-3)^{-k}}{2k+1}$$

$$\pi = \cfrac{4}{1+\cfrac{1^2}{2+\cfrac{3^2}{2+\cfrac{5^2}{\cdots}}}}$$

$$\frac{\pi}{2}=\prod_{n=1}^{\infty} \frac{(2n)^2}{(2n-1)(2n+1)}$$

$$\pi = 2+\cfrac{2}{1+\cfrac{1\cdot 2}{1+\cfrac{2\cdot 3}{1+\cfrac{3\cdot 4}{1+\cdots}}}}$$

$$\frac{5}{4} - \zeta(3) = \sum_{n=2}^{\infty} \frac{1}{n^3(n^2-1)}$$

Our Goal – *I*

Fast Convergence

Srinivasa Ramanujan

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{k=0}^{\infty} \frac{(4k)!\,(1103 + 26390k)}{(k!)^4 396^{4k}}$$

- M. Shalyt et al. Unifying Structure for Ramanujan's 17 Series for $1/\pi$: A Human-AI Discovery. *RISC Proceedings on Symbolic Computation and Machine Learning*, 3:82-83, 2026.
- R. Hemmecke et al. Computer-Assisted Construction of Ramanujan-Sato Series for 1 over π. *RISC Report Series 25-01*, 2025.
- S. Ramanujan. Modular equations and approximations to π. *Quarterly Journal of Mathematics,* 45:350-372, 1914.

Our Goal – *II*

Irrationality

Roger Apéry

$$(n+1)^3 x_{n+1} = \left(34n^3 + 51n^2 + 27n + 5\right)x_n - n^3 x_{n-1}$$

Take:

$$A_0 = 1,\ A_1 = 5 \quad \text{and} \quad B_0 = 0,\ B_1 = 6$$

Then:

$$B_n - \zeta(3)A_n \sim \left(\sqrt{2} - 1\right)^{4n}$$

Hence:

$$\frac{A_n}{B_n} \to \zeta(3)$$

R. Apéry. Irrationalit´e de $\zeta(2)$ et $\zeta(3)$. *Astérisque*, 61:11–13, 1979.

Approximations & Irrationality

Define Irrationality measure of an approximation:

$$\boxed{\delta = \lim_{n\to\infty} \delta_n\,, \qquad \delta_n = -1 - \frac{\log\left|x - \frac{p_n}{q_n}\right|}{\log|q_n|}}$$

In Apéry's case:

$$p_n = \frac{A_n}{d_n}, \qquad q_n = \frac{B_n}{d_n}, \qquad d_n = \mathrm{lcm}(1, \dots, \mathrm{n})^{-3} \sim e^{-3n}$$

Then:

$$\left|\frac{p_n}{q_n} - \zeta(3)\right| < \frac{1}{|q_n|^{1+\delta}} \text{ where } \delta = 0.0805 > 0 \Longrightarrow \zeta(3) \notin \mathbb{Q}$$

$\Longrightarrow$ **We want to optimize for $\boldsymbol{\delta}$!**

R. Apéry. Irrationalit´e de $\zeta(2)$ et $\zeta(3)$. *Astérisque*, 61:11–13, 1979.

Linear Recurrences

$$u_n = a_{1,n}u_{n-1} + a_{2,n}u_{n-2} + \cdots + a_{m,n}u_{n-m}$$

General form:

$$u_n = a_n u_{n-1} + b_n u_{n-2}$$

a_n	b_n	Limit
$n^3 + (n+1)^3$	$-n^6$	$\zeta(3)^{-1}$
$n^3 + (n+1)^3 + 4(2n+1)$	$-n^6$	$(\zeta(3) - 1)^{-1}$
$3n^2 + 3n + 1$	$-2n^4$	$(2G)^{-1}$
$n^3 + (n+1)^3 + (n+1)^2$	$n^5(n+1)$	$(\zeta(3) - \zeta(2) + 1)^{-1}$

Take solutions: $p_{-1} = 1, p_0 = a_0, q_{-1} = 0, q_0 = 1$ *then* $\frac{p_n}{q_n} \to L$

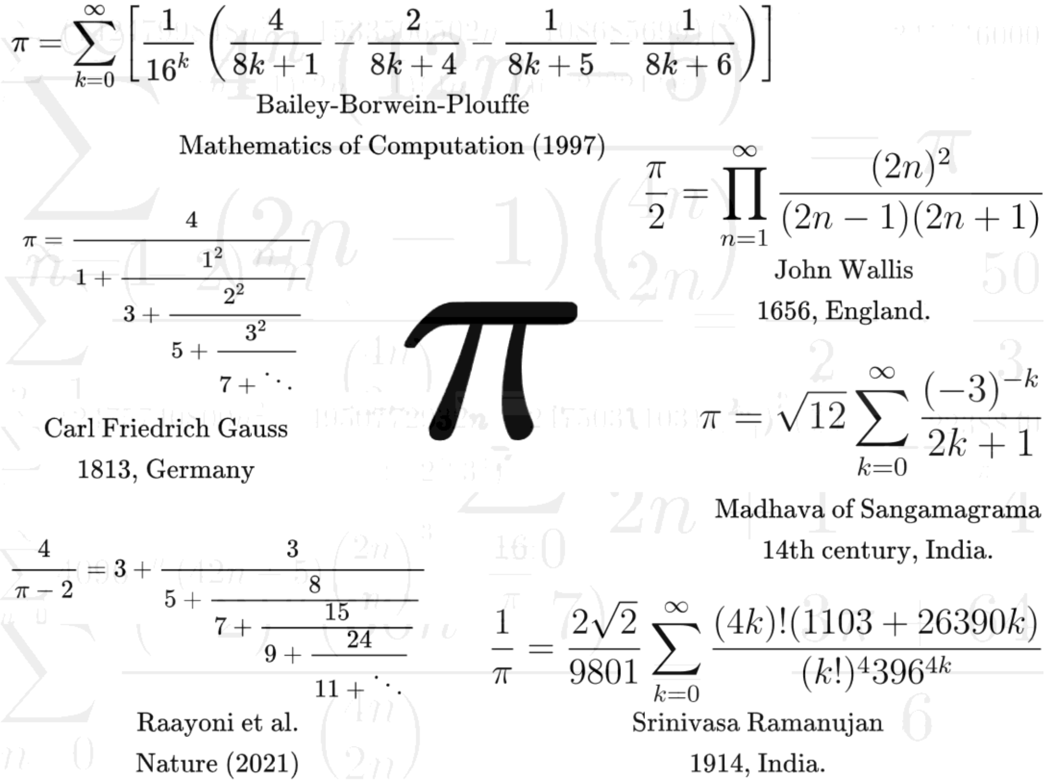
R. Elimelech et al. Algorithm-assisted discovery of an intrinsic order among mathematical constants. *Proceedings of the National Academy of Sciences*, 121(25):e2321440121, 2024.

What was found

Unify formulas for π

$+$

Connect D-finite functions



T. Raz et al. From Euler to AI: Unifying formulas for mathematical constants. In *Advances in Neural Information Processing Systems*, 2025.
S. Weinbaum et al. On conservative matrix fields: Continuous asymptotics and arithmetic, 2025.

What is a CMF?

$$M_Y(x, y)M_X(x, y+1) = M_X(x, y)M_Y(x+1, y)$$

$$\begin{tikzcd}
(x, y+1) \arrow[rr, "M_X(x,y+1)"] & & (x+1, y+1) \\
(x, y) \arrow[u, dashed, "M_Y(x,y)"] \arrow[rr, "M_X(x,y)"'] \arrow[urr, phantom, "\circlearrowleft"] & & (x+1, y) \arrow[u, dashed, "M_Y(x+1,y)"']
\end{tikzcd}$$

R. Elimelech et al. Algorithm-assisted discovery of an intrinsic order among mathematical constants. *Proceedings of the National Academy of Sciences*, 121(25):e2321440121, 2024.

What is a CMF?

Example – CMF from a Generalized Hypergeometric:

$$pFq\begin{bmatrix} x_1 & \dots & x_p \\ x_{p+1} & \dots & x_{p+q} \end{bmatrix}; z\Big] = \sum_{n=0}^{\infty} \frac{(x_1)_n \dots (x_p)_n}{(x_{p+1})_n \dots (x_{p+q})_n} \frac{z^n}{n!}$$

From ODE and contiguous relations → $M_{\theta_z}^{pFq}$ (Companion Matrix):

$$\forall 1 \leq i \leq p: \; M_{e_i} = \frac{1}{x_i} M_{\theta_z}^{pFq} + I,$$

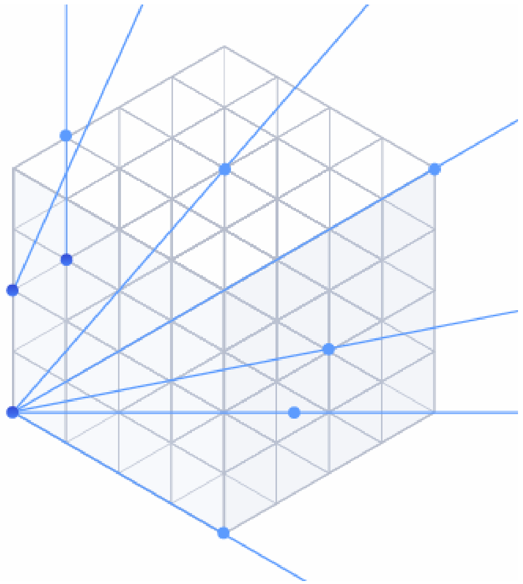
$$\forall p+1 \leq i \leq p+q: \; M_{-e_j} = \frac{1}{x_i - 1} M_{\theta_z}^{pFq} + I$$

$$\boxed{(a)_n := \prod_{k=0}^{n-1} (a+k)}$$

S. Weinbaum et al. On conservative matrix fields: Continuous asymptotics and arithmetic, 2025.

CMF Limit

So how do we get our $\frac{p_n}{q_n}$ *actually?*



Define CMF limit as:

$$\mathcal{L}_{x,v}^{p',q'}(n) = \frac{p'^T \cdot \mathcal{M}_{nv}(x) \cdot e_r}{q'^T \cdot \mathcal{M}_{nv}(x) \cdot e_r} = \frac{p_n}{q_n}$$

Where:

- $v :=$ trajectory
- $x :=$ start point
- $\mathcal{M}_{nv}(x) :=$ "walk" matrix starting from x and walking up to $x + nv$.

And how do we get p', q'*?* - PSLQ

S. Weinbaum et al. On conservative matrix fields: Continuous asymptotics and arithmetic, 2025.

And Practically?

Create and interact with CMFs

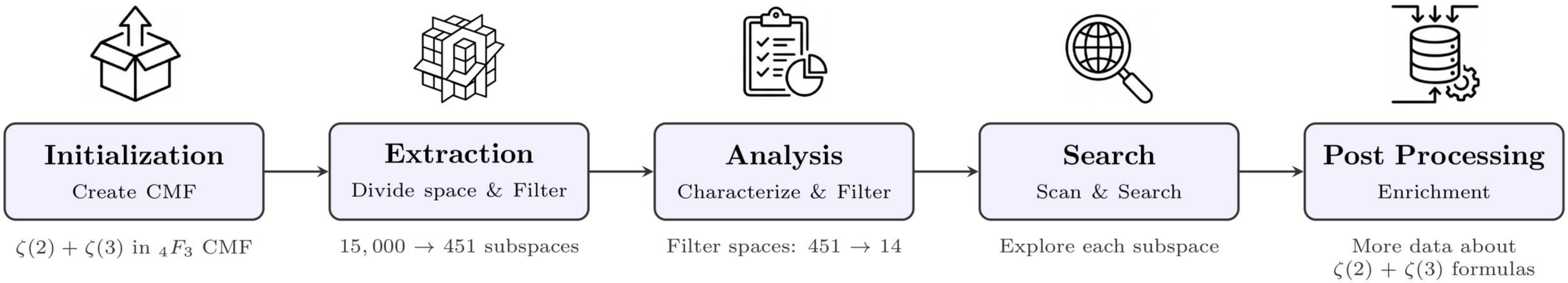
Ramanujan Tools
By Rotem Kalisch
(and others)

$\Longrightarrow$

Search and explore new formulas

Ramanujan's Dreams
By Me!

Our Solution - Overview



Reminder: What do we want to achieve?

- Search for hints of irrationality
- Recurrence relation data mining

Creation

Example: ${}_2F_1(z=-1)$ searching for $log(2)$.

$$CM := \begin{pmatrix} 0 & -\frac{xy}{2} \\ 1 & -\frac{x+y+z-1}{2} \end{pmatrix}$$

Hence:

$$M_x := \begin{pmatrix} 0 & -\frac{y}{2} \\ \frac{1}{x} & \frac{x-y-z+1}{2x} \end{pmatrix}, \qquad M_y := \begin{pmatrix} 0 & -\frac{x}{2} \\ \frac{1}{y} & \frac{-x+y-z+1}{2y} \end{pmatrix}, \qquad M_z := \frac{z}{(x-z)(y-z)} \begin{pmatrix} -x-y+z & xy \\ -2 & 2z \end{pmatrix}$$

Creation

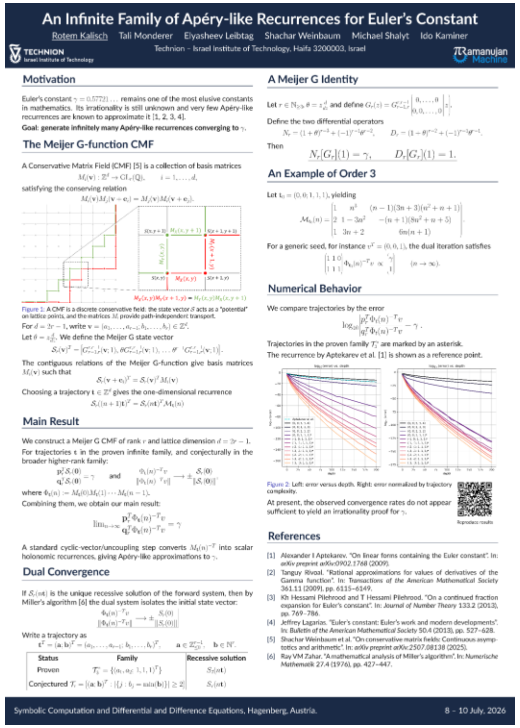
$$M_x \coloneqq \begin{pmatrix} 0 & -\frac{y}{2} \\ \frac{1}{x} & \frac{x-y-z+1}{2x} \end{pmatrix}$$

$$M_y \coloneqq \begin{pmatrix} 0 & -\frac{x}{2} \\ \frac{1}{y} & \frac{-x+y-z+1}{2y} \end{pmatrix}$$

$$M_z \coloneqq \frac{z}{(x-z)(y-z)} \begin{pmatrix} -x-y+z & xy \\ -2 & 2z \end{pmatrix}$$

Hypergeometric

An Infinite Family of Apéry-like Recurrences of Euler's Constant



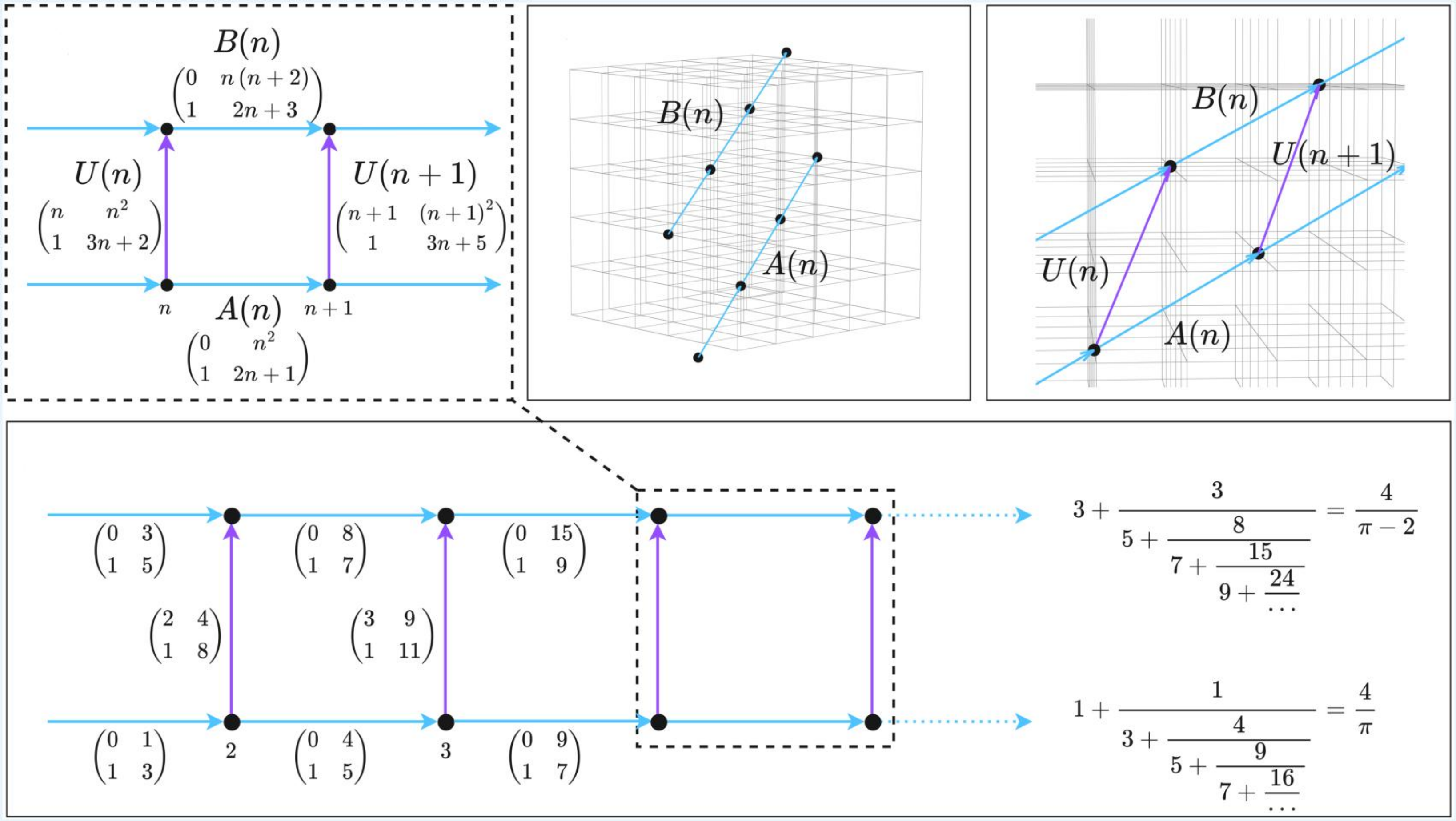
Rotem Kalisch

Meijer G

Extraction

<u>Require</u>

1. Want to use full space – step in negative directions.
2. Use a single start point.



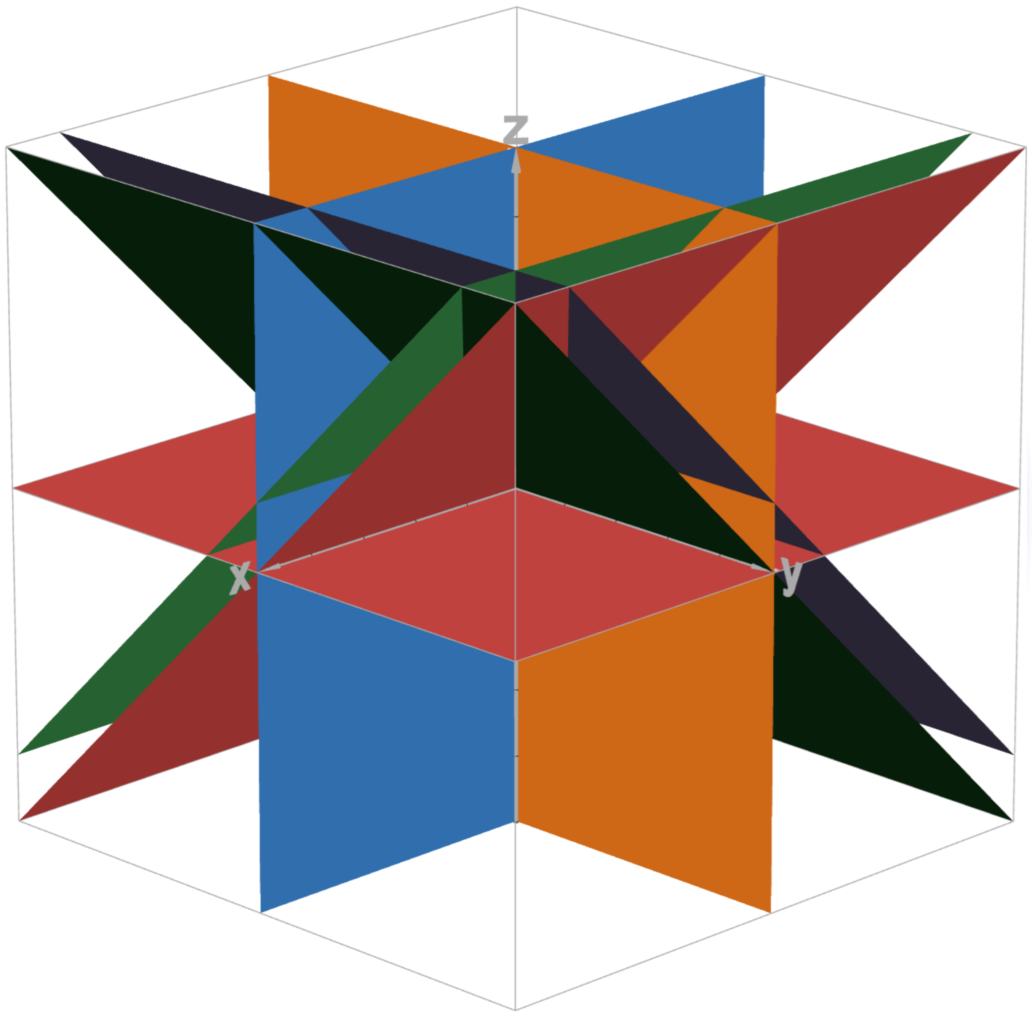
Extraction

<u>CMF "walls":</u>

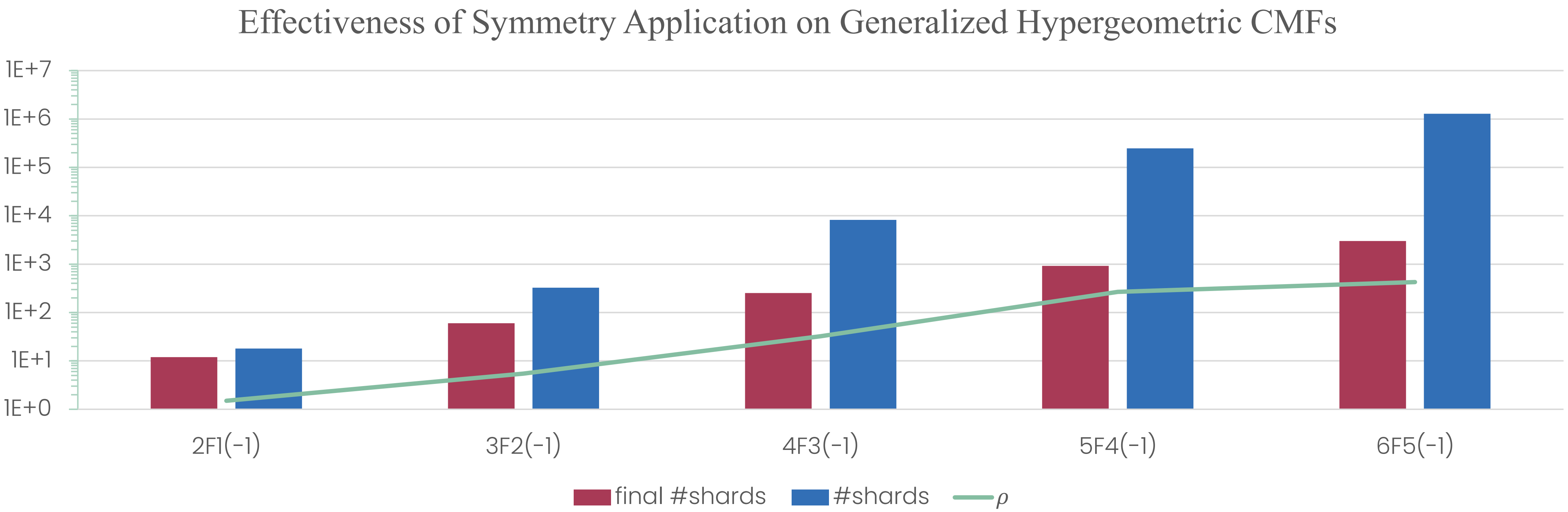
$$x = 0, \qquad x - z = 0, \qquad x - z = -1$$
$$y = 0, \qquad y - z = 0, \qquad y - z = -1$$
$$z = 0$$
$$\downarrow$$
$$A\boldsymbol{x} = b$$

Shard: Unbounded chamber $A'x < b$ defined by the hyperplanes $:= \{M_i\text{'s poles } \cup \det(M_i)\text{'s zeros}\}$.

Goal: find all shards in the CMF



Symmetry Shard Reduction



$$\rho = \frac{\#shards}{\text{fianl } \#shards}$$

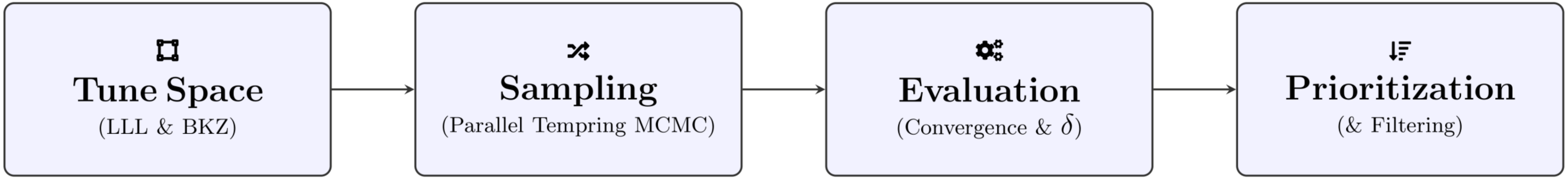
Analysis

- Want trajectory v s.t. v is a recession direction.

  i.e.: v stays in a shard defined by (A, b) if $Av \leq 0$

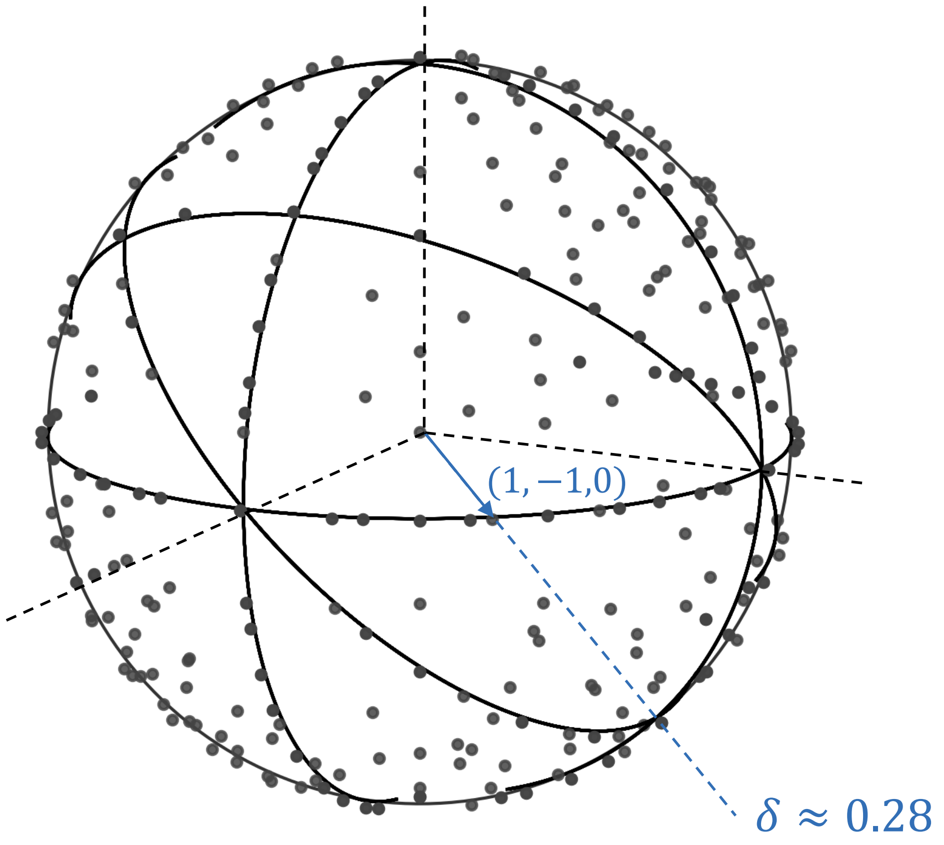
- Which Shards are interesting?

  Check convergence to constant and other attributes → sample and scan!



Analysis

$$-(n+2)(2n+5)u_n = 12(n+3)^2 u_{n-1} - (n+4)(2n+7)u_{n-2}$$



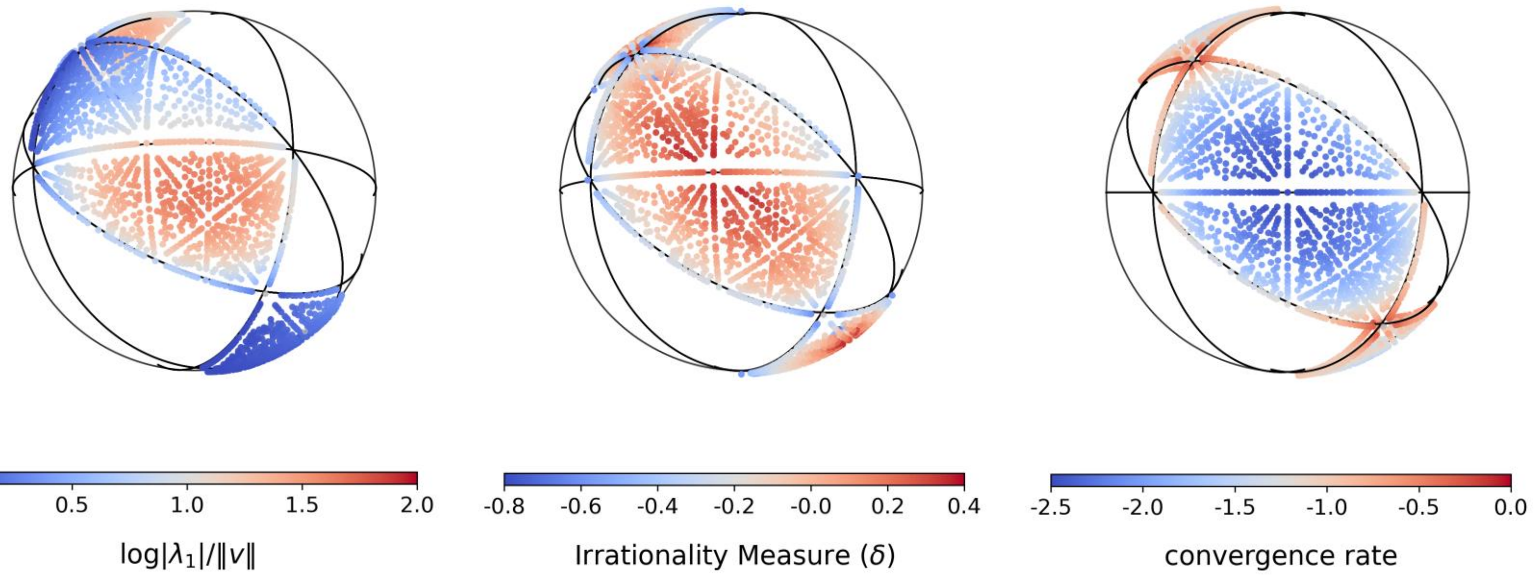
Search

Methods:

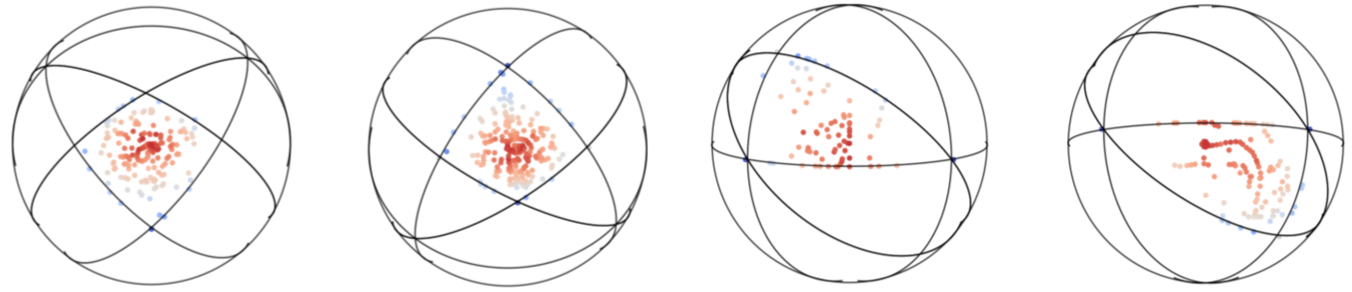
- Broad scan
- Simulated Annealing
- Genetic algorithm

Utilizing continuity:

- Gradient Ascent



Gradient Ascent for δ Search – SPSA and Adam

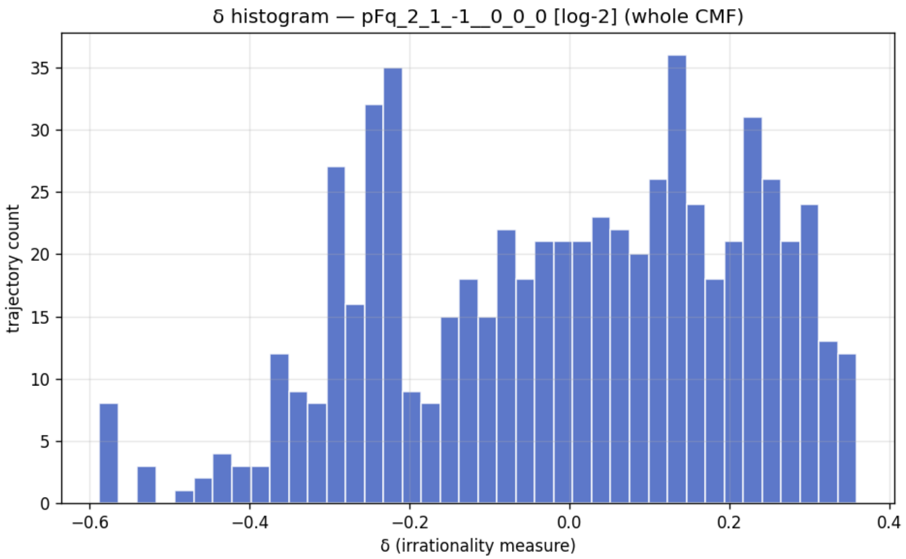


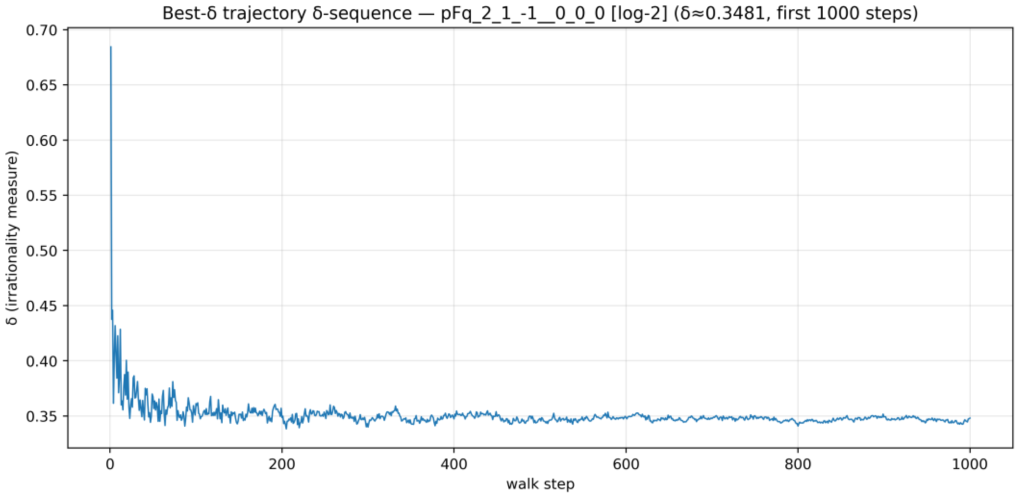
Continutiy conjectures: S. Weinbaum et al. On conservative matrix fields: Continuous asymptotics and arithmetic, 2025.

Post Processing

- Attributes:
  - Symbolic formula representation
  - Digits per step
  - $\gcd(p_n, q_n)$ slope
  - Convergence rate
  - Eigen values

  …
- Graphs (statistics)





Findings

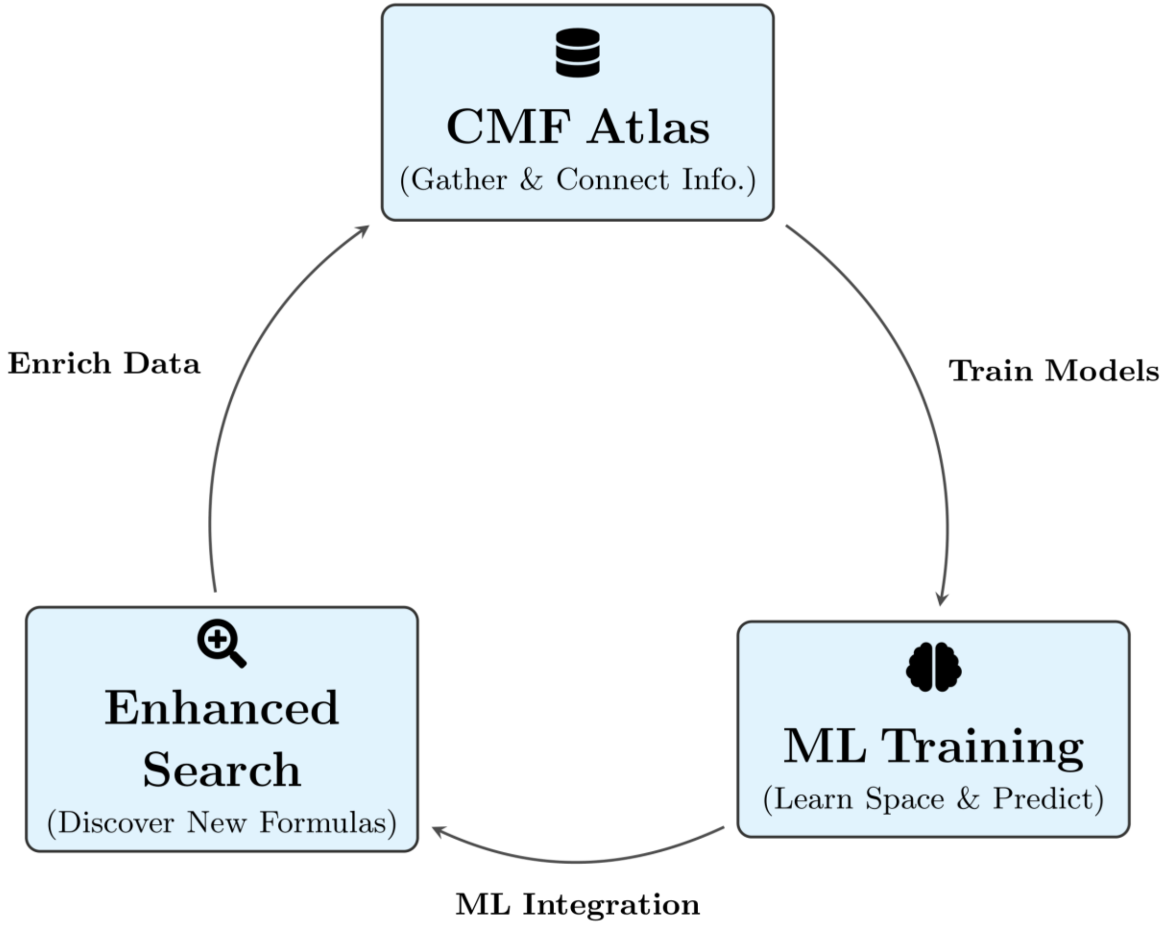
Constant	CMF	δ	Dimensions
G	${}_4F_3(\mathbf{a},\mathbf{b};1)$	-0.275	7
$\zeta(2)$	${}_4F_3(\mathbf{a},\mathbf{b};1)$	0.26	7
$\zeta(2)+\zeta(3)$	${}_4F_3(\mathbf{a},\mathbf{b};1)$	$-0.0002\ (\pm 0.001)$	7
$\frac{4}{3\sqrt{3}}\mathcal{G}_{Gi}$	${}_3F_2\left(\mathbf{a},\mathbf{b};\frac{1}{4}\right)$	-0.06	5
π	integral	0.16	3
$\log(2)$	${}_2F_1\left(\mathbf{a},\mathbf{b};-1\right)$	0.346	3
$\log(2)$	${}_3F_2\left(\mathbf{a},\mathbf{b};\frac{1}{2}\right)$	0.388	5
G	${}_3F_2\left(\mathbf{a},\mathbf{b};1\right)$	-0.43	5

WR δ improvement

WR δ reconstruction

Checkout: Hila Barkan – “New Directions in Irrationality”

Vision and Future



Thanks ☺

More details at

- *Weinbaum et al.,* On Conservative Matrix Fields: Continuous Asymptotics and Arithmetic. *2025*.
- *Raz et al.,* From Euler to AI: Unifying Formulas for Mathematical Constants. *In Advances in Neural Information Processing Systems, 2025*.
- *Elimelech et al.,* Algorithm-assisted discovery of an intrinsic order among mathematical constants. *Proceedings of the National Academy of Sciences, 2024*.