SCDDE July 8th, 20026 (the day joint with SCML)
RISC, Johannes Kepler University Linz

# The Unreasonable Effectiveness of Computer Algebra in the Mathematical Sciences

Peter Paule

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# Matching data to knowledge

# Informal definition

To match some data $A$ to $B$ means to find some $B$ from some knowledge base that captures relevant aspects of $A$.

# A signal (from aliens?):

Out[0]= 0.318309886183790671537767526745028724068
919291480912897495334688117793595268453070180227605532506171912145685453515916073785823692229157305755934821463399678458479933874818155146155492793850615377434 79...

# A signal (from aliens?):

Out[1]= 0.31830988618379067153776752674502872406 8
91929148091289749533468811779359526845 30
70180227605532506171912145685453515916 07
37858236922291573057559348214633996784 58
47993387481815514615549279385061537743 479...

A value produced by a function?

In[2]:= $\mathbf{N}[\mathbf{R[25]}, \mathbf{200}]$

Out[2]= 0.31830988618379067153776752674502872406 8
91929148091289749533468811779359526845 30
70180227605532506171912145685453515916 07
37858236922291573057559348214633996784 58
47993387481815514615549279385061537743 479

In[3]:= $\mathbf{R[m\_] := \frac{\sqrt{8}}{9801} \sum_{n=0}^{m} \frac{(4n)!}{(n!)^4} \frac{1103 + 26390n}{396^{4n}}}$

In[5]:= $\mathbf{R[m\_]} := \frac{\sqrt{8}}{9801} \sum_{n=0}^{m} \frac{(4n)!}{(n!)^4} \frac{1103 + 26390n}{396^{4n}}$

In[6]:= $\{\mathbf{R[0]}, \mathbf{N[R[0], 8]}\}$

In[7]:= $\mathbf{R[m\_] := \frac{\sqrt{8}}{9801} \sum_{n=0}^{m} \frac{(4n)!}{(n!)^4} \frac{1103 + 26390n}{396^{4n}}}$

---

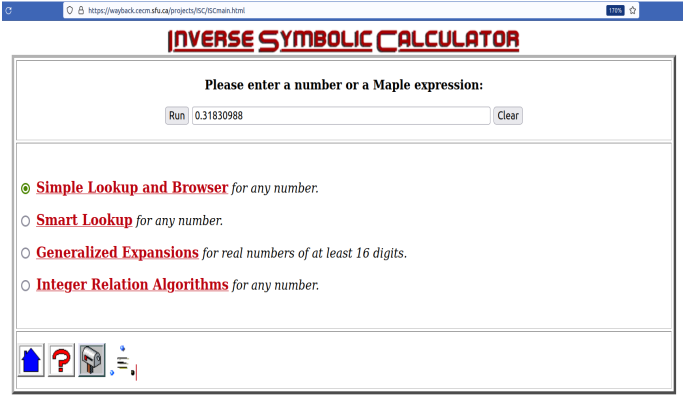
In[8]:= $\mathbf{\{R[0], N[R[0], 8]\}}$

Out[8]= $\{\frac{2206\,\mathtt{Sqrt}[2]}{9801}, 0.31830988\}$

In[9]:= $\mathbf{R[m\_]} := \dfrac{\sqrt{8}}{9801} \sum_{n=0}^{m} \dfrac{(4n)!}{(n!)^4} \dfrac{1103 + 26390n}{396^{4n}}$

In[10]:= $\{\mathbf{R[0]}, \mathbf{N[R[0], 8]}\}$

Out[10]= $\left\{\dfrac{2206\,\mathrm{Sqrt}[2]}{9801}, 0.31830988\right\}$



INVERSE SYMBOLIC CALCULATOR

# Results of the search:

```
        Your input of .31830988 was probably generated by one
        the following functions or found in one of the given tables.
        Answers are given from shortest to longest description
```

**Bases constants, Pi, e, sqrt(2), etc...**
3183098861837906 = 1/Pi

**Mixed constants with 5 operations**
3183098861837906 = 1/Pi

**Mixed constants with 5 operations**
3183098861837906 = (1/2)^(ln(Pi)/ln(2))

**Mixed constants with 5 operations**
3183098861837906 = (1/3)^(ln(Pi)/ln(3))

**Mixed constants with 5 operations**
3183098861837906 = sr(2)/GAM(1/4)/GAM(3/4)

**Sum(1/Product(A(k),k=1..n),n=1..inf), Axxxxxx is from Enc. Integer S$**
3183098861837906 = Sum(1/prod(A014012(k),k=1..n),n=1..inf)

## Browse around .31830988.

In[11]:= $$\mathbf{R[m\_] := \frac{\sqrt{8}}{9801} \sum_{n=0}^{m} \frac{(4n)!}{(n!)^4} \frac{1103 + 26390n}{396^{4n}}}$$

In[14]:= $\mathbf{R[m\_] := \frac{\sqrt{8}}{9801} \sum_{n=0}^{m} \frac{(4n)!}{(n!)^4} \frac{1103 + 26390n}{396^{4n}}}$

---

In[15]:= $\mathbf{N[\{1/Pi, R[0]\}]}$

In[17]:= $\mathbf{R[m\_] := \frac{\sqrt{8}}{9801} \sum_{n=0}^{m} \frac{(4n)!}{(n!)^4} \frac{1103 + 26390n}{396^{4n}}}$

---

In[18]:= $\mathbf{N[\{1/Pi, R[0]\}]}$

Out[18]= $\{0.31831, 0.31831\}$

In[19]:= $\mathbf{N[\{1/Pi, R[0]\}, 50]}$

In[20]:= $\mathbf{R[m\_] := \frac{\sqrt{8}}{9801} \sum_{n=0}^{m} \frac{(4n)!}{(n!)^4} \frac{1103 + 26390n}{396^{4n}}}$

In[21]:= $\mathbf{N[\{1/Pi, R[0]\}]}$

Out[21]= $\{0.31831, 0.31831\}$

In[22]:= $\mathbf{N[\{1/Pi, R[0]\}, 50]}$

Out[22]= $\{0.318309886183790671537767526745028724068919291480912,$

$0.318309878440470123217684453178919902185970427209630\}$

In[23]:= $\mathbf{R[m\_] := \frac{\sqrt{8}}{9801} \sum_{n=0}^{m} \frac{(4n)!}{(n!)^4} \frac{1103 + 26390n}{396^{4n}}}$

---

In[24]:= $\mathbf{N[\{1/Pi, R[0]\}]}$

Out[24]= $\{0.31831, 0.31831\}$

In[25]:= $\mathbf{N[\{1/Pi, R[0]\}, 50]}$

Out[25]= {0.31830988618379067153776752674502872406891929148091,
0.31830987844047012321768445317891990218597042720963}

---

In[26]:= $\mathbf{N[\{1/Pi, R[1]\}, 50]}$

Out[26]= {0.31830988618379067153776752674502872406891929148091,
0.31830988618379060673919643382082822511793407438051}

---

In[27]:= $\mathbf{R[m\_] := \frac{\sqrt{8}}{9801} \sum_{n=0}^{m} \frac{(4n)!}{(n!)^4} \frac{1103 + 26390n}{396^{4n}}}$

---

In[28]:= $\mathbf{N[\{1/Pi, R[0]\}]}$

Out[28]= $\{0.31831, 0.31831\}$

In[29]:= $\mathbf{N[\{1/Pi, R[0]\}, 50]}$

Out[29]= {0.31830988618379067153776752674502872406891929148091,
0.31830987844047012321768445317891990218597042720963}

---

In[30]:= $\mathbf{N[\{1/Pi, R[2]\}, 50]}$

Out[30]= {0.31830988618379067153776752674502872406891929148091,
0.31830988618379067153776695099517846798758674972806}

$$R[\infty] = 2\sqrt{2}\sum_{n=0}^{\infty}\frac{(1/4)_n(1/2)_n(3/4)_n}{(1)_n(1)_n}(1103+26390n)\frac{(1/99)^{4n+2}}{n!} \overset{(44)}{=} \frac{1}{\pi}.$$

---

$(a)_n = a(a+1)\dots(a+n-1), n \geq 1$ and $(a)_0 = 1.$

$$R[\infty] = 2\sqrt{2}\sum_{n=0}^{\infty}\frac{(1/4)_n(1/2)_n(3/4)_n}{(1)_n(1)_n}(1103+26390n)\frac{(1/99)^{4n+2}}{n!}$$
$$\stackrel{(44)}{=} \frac{1}{\pi}.$$

---

$$(a)_n = a(a+1)\ldots(a+n-1), n \geq 1 \quad \text{and} \quad (a)_0 = 1.$$

> Ramanujan (1914): "The last series (44) is extremely rapidly convergent. Thus, taking only the first term, we see that..."

$$R[\infty] = 2\sqrt{2}\sum_{n=0}^{\infty} \frac{(1/4)_n(1/2)_n(3/4)_n}{(1)_n(1)_n}(1103 + 26390n)\frac{(1/99)^{4n+2}}{n!} \overset{(44)}{=} \frac{1}{\pi}.$$

---

$$(a)_n = a(a+1)\dots(a+n-1), n \geq 1 \quad \text{and} \quad (a)_0 = 1.$$

Ramanujan (1914): "The last series (44) is extremely rapidly convergent. Thus, taking only the first term, we see that..."

Ramanujan (1914) presented $17$ series for $1/\pi$ without proof and with only scanty motivation.[1]

[1] J. Borwein & P. Borwein & D. Bailey, "Ramanujan, modular equations, and approximations to pi, or How to compute one billion digits of pi" (1989).

- Ramanujan's (44) series,

$$\frac{1}{\pi} \overset{(44)}{=} 2\sqrt{2}\sum_{n=0}^{\infty} \frac{(1/4)_n(1/2)_n(3/4)_n}{(1)_n(1)_n}(1103+26390n)\frac{(1/99)^{4n+2}}{n!}.$$

In 1985 Bill Gosper used (44) to compute 17,526,100 digits of $\pi$, which at that time was a world record.

- Ramanujan's (44) series,

$$\frac{1}{\pi} \overset{(44)}{=} 2\sqrt{2}\sum_{n=0}^{\infty} \frac{(1/4)_n(1/2)_n(3/4)_n}{(1)_n(1)_n}(1103+26390n)\frac{(1/99)^{4n+2}}{n!}.$$

In 1985 Bill Gosper used (44) to compute 17,526,100 digits of $\pi$, which at that time was a world record.

**REMARK 1. There was only one problem with his calculation:**

- Ramanujan's (44) series,

$$\frac{1}{\pi} \overset{(44)}{=} 2\sqrt{2}\sum_{n=0}^{\infty} \frac{(1/4)_n(1/2)_n(3/4)_n}{(1)_n(1)_n}(1103+26390n)\frac{(1/99)^{4n+2}}{n!}.$$

In 1985 Bill Gosper used (44) to compute 17,526,100 digits of $\pi$, which at that time was a world record.

**REMARK 1. There was only one problem with his calculation: (44) had not yet been proven!**

---

NOTE. Nowadays even Mathematica knows,

In[33]:= **R[Infinity]**

Out[33]= $\frac{1}{\pi}$

- Ramanujan's (44) series,

$$\frac{1}{\pi} \overset{(44)}{=} 2\sqrt{2}\sum_{n=0}^{\infty} \frac{(1/4)_n(1/2)_n(3/4)_n}{(1)_n(1)_n}(1103+26390n)\frac{(1/99)^{4n+2}}{n!}.$$

In 1985 Bill Gosper used (44) to compute 17,526,100 digits of $\pi$, which at that time was a world record.

**REMARK 2. In 1988 Borwein & Borwein gave a "proof" of (44) based on sophisticated math & confidence in numerics.**

---

- Ramanujan's (44) series,

$$\frac{1}{\pi} \stackrel{(44)}{=} 2\sqrt{2}\sum_{n=0}^{\infty} \frac{(1/4)_n(1/2)_n(3/4)_n}{(1)_n(1)_n}(1103+26390n)\frac{(1/99)^{4n+2}}{n!}.$$

In 1985 Bill Gosper used (44) to compute 17,526,100 digits of $\pi$, which at that time was a world record.

**REMARK 2. In 1988 Borwein & Borwein gave a "proof" of (44) based on sophisticated math & confidence in numerics.**

---

They used the fact that the first 10,000,000 digits of $\pi$ computed by Gosper coincided with those computed by Yasumasa Kanada et al. in 1982. Given this accuracy, a numerical estimation implied the integer **1103** being the correct coefficient.

- Ramanujan's (44) series,

$$\frac{1}{\pi} \overset{(44)}{=} 2\sqrt{2}\sum_{n=0}^{\infty}\frac{(1/4)_n(1/2)_n(3/4)_n}{(1)_n(1)_n}(1103+26390n)\frac{(1/99)^{4n+2}}{n!}.$$

In 1985 Bill Gosper used (44) to compute 17,526,100 digits of $\pi$, which at that time was a world record.

**REMARK 3.** Today Ramanujan's identity (44) can be discovered and proven algorithmically; this includes a purely "algebraic" proof of $1103$ being the correct integer value:

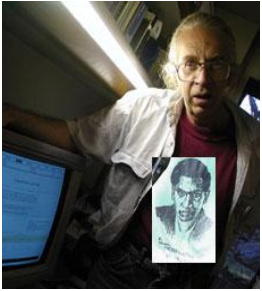
Hemmecke & P. & Radu: Computer-assisted construction of Ramanujan–Sato series for $1$ over $\pi$, Ramanujan J., 2026.

- Ramanujan's (44) series,

$$\frac{1}{\pi} \overset{(44)}{=} 2\sqrt{2} \sum_{n=0}^{\infty} \frac{(1/4)_n (1/2)_n (3/4)_n}{(1)_n (1)_n} (1103 + 26390n) \frac{(1/99)^{4n+2}}{n!}.$$

In 1985 Bill Gosper used (44) to compute 17,526,100 digits of $\pi$, which at that time was a world record.

**REMARK 4.** Needless to say, in a jiffy, Gosper would have identified the signal of the aliens as coming from Ramanujan's formula!
(Picture: `https://gosper.org/bill.html`)



# “Dictionaries”

# A Dictionary of Real Numbers

**Jonathan Borwein**
*Dalhousie University*

**Peter Borwein**
*Dalhousie University*

*Wadsworth & Brooks/Cole Advanced Books & Software*
*Pacific Grove, California*

## From the Preface (1990):

How do we recognize that the number .93371663... is actually $2\log_{10}(e+\pi)/2$ ? Gauss observed that the number 1.85407467... is (essentially) a rational value of an elliptic integral—an observation that was critical in the development of nineteenth century analysis. How do we decide that such a number is actually a special value of a familiar function without the tools Gauss had at his disposal, which were, presumably, phenomenal insight and a prodigious memory? Part of the answer, we hope, lies in this volume.

This book is structured like a reverse telephone book, or more accurately, like a reverse handbook of special function values. It is a list of just over 100,000 eight-digit real numbers in the interval $[0,1)$ that arise as the first eight digits of special values of familiar functions. It is designed for people, like ourselves,

## Matching our 8 digits to $\frac{1}{\pi}$:

3167 8992 ln : $3/4+4\sqrt{5}/3$
3168 0150 $t\pi$ : $3\sqrt{2}+2\sqrt{3}$
3168 0574 cu : $4\sqrt{3}/3$
3168 1214 $t\pi$ : $2\sqrt{2}/3-\sqrt{5}$
3168 1410 $J_1$ : $\exp(\sqrt{5}/2)$
3168 1460 rt : (3,8,2,-1)
3168 2130 cu : $1+\sqrt{5}/3$
3168 3545 $\exp(e)\div\exp(3/5)$
3168 5042 rt : (2,4,8,-3)
3168 5740 $E$ : $\ln(\sqrt{2}/3)$
3168 6811 rt : (8,1,-8,6)
3168 7956 $2^x$ : $6\sqrt{2}/7$
3168 8446 at : $3\sqrt{2}/2+\sqrt{3}$
3168 9001 $J_1$ : $1/\ln(\sqrt[3]{3}/2)$
3168 9382 $J_1$ : $3\sqrt{5}/10$
3169 0804 rt : (5,2,-3,4)

3173 4531 $2^x$ : $3/4+3\sqrt{2}/2$
3173 4942 rt : (4,9,-7,-3)
3173 5261 rt : (3,0,6,2)
3173 6155 sr : $9\pi$
3173 7051 ln : $2e+3\pi/2$
3173 7299 $2/3\div\ln(3/4)$
3173 7386 rt : (2,9,-7,9)
3173 7619 sr : $P_{61}$
3173 7849 sq : $2/3+2\sqrt{3}/3$
3173 9380 $\ln(3/5)\div\ln(5)$
3174 1055 rt : (7,1,-4,9)
3174 2435 $t\pi$ : $4\sqrt{2}-\sqrt{5}/4$
3174 3541 id : $2\sqrt{3}/3-2\sqrt{5}$
3174 5115 rt : (5,2,-6,-7)
3174 5485 $E$ : $5\zeta(3)/8$
3174 5688 rt : (6,6,-5,-2)

3178 1165 rt : (9,1,-9,7)
3178 1589 rt : (7,0,-7,2)
3178 1675 $\Psi$ : $(\sqrt[3]{3}/2)^{-1/2}$
3178 2370 as : $5/16$
3178 2848 $\Psi$ : $(2\pi)^{1/3}$
3178 3724 id : $\sqrt{2}-\sqrt{3}$
3178 4300 cr : $1/4+3e/4$
3178 6426 $J$ : $\ln 2/8$
3178 7378 id : $8e\pi$
3178 8717 rt : (4,2,1,7)
3178 8931 cu : $2\sqrt{2}-\sqrt{3}$
3179 3078 $e^x$ : $e-\pi/3$
3179 3128 rt : (2,6,0,7)
3179 4452 $J_1$ : $23/4$
3179 4671 $\ell_{10}$ : $3\ln 2$
3179 5082 $2^x$ : $(e/2)^{-3}$

3182 9512 cr : $\sqrt{3}+\sqrt{5}/4$
3182 9742 $\Gamma$ : $11/16$
3182 9910 cu : $\sqrt{2}/4-3\sqrt{5}/4$
3183 0988 id : $(1/\pi)$
3183 3588 rt : (4,8,-1,-1)
3183 4907 $\ell_{10}$ : $(\ln 2)^2$
3183 7338 rt : (2,7,0,-4)
3183 7771 rt : (9,7,0,-1)
3183 8392 $t\pi$ : $3\sqrt{2}/4-3\sqrt{5}/2$
3183 9039 $Ei$ : $5e\pi/9$
3184 1104 $\ell_{10}$ : $\sqrt{13/3}$
3184 1223 rt : (6,2,-4,5)
3184 3476 $10^x$ : $3/2+3\sqrt{5}/4$
3184 4428 $\ln(4)\times s\pi(2/5)$
3184 4433 $e^x$ : $(\sqrt{2})^{-1/2}$
3184 5373 ln : $8/11$

# Another dictionary (1990):

THE
ENCYCLOPEDIA
OF
INTEGER
SEQUENCES

**N. J. A. Sloane**
*Mathematical Sciences Research Center*
*AT&T Bell Laboratories*
*Murray Hill, New Jersey*

**Simon Plouffe**
*Départment de Mathématiques et d'Informatique*
*Université du Québec à Montréal*
*Montréal, Québec*

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## ONLINE version:

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founded in 1964 by N. J. A. Sloane

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Enter a sequence, word, or sequence number:

3,1,8,3,0,9,8,8

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oeis.org/Seis.html

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3,1,8,3,0,9,8,8 Search Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

Search: **seq:3,1,8,3,0,9,8,8**

Displaying 1-1 of 1 result found. page 1

Sort: relevance | references | number | modified | created Format: long | short | data

A049541 Decimal expansion of 1/Pi. +30 85

**3, 1, 8, 3, 0, 9, 8, 8**, 6, 1, 8, 3, 7, 9, 0, 6, 7, 1, 5, 3, 7, 7, 6, 7, 5, 2, 6, 7, 4, 5, 0, 2, 8, 7, 2, 4, 0, 6, 8, 9, 1, 9, 2, 9, 1, 4, 8, 0, 9, 1, 2, 8, 9, 7, 4, 9, 5, 3, 3, 4, 6, 8, 8, 1, 1, 7, 7, 9, 3, 5, 9, 5, 2, 6, 8, 4, 5, 3, 0, 7, 0, 1, 8, 0, 2, 2, 7, 6, 0, 5, 5, 3, 2, 5, 0, 6, 1, 7, 1 (list; constant; graph; refs; listen; history; text; internal format)

OFFSET 0,1

COMMENTS The ratio of the volume of a regular octahedron to the volume of the circumscribed sphere (which has circumradius a*sqrt(2)/2 = a*A010503, where a is the octahedron's edge length; see MathWorld link). For similar ratios for other Platonic solids, see A165922, A165952, A165953 and A165954. - Rick L. Shepherd, Oct 01 2009

Corresponds to a gauge point marked "M" on slide rule calculating devices in the 20th century. The Pickworth reference notes its use in calculating the area of the curved surface of a cylinder. - Peter Munn, Aug 14 2020

REFERENCES J.-P. Delahaye, Pi - die Story (German translation), Birkhäuser, 1999 Baasel, p. 245. French original: Le fascinant nombre Pi, Pour la Science, Paris, 1997.

C. N. Pickworth, The Slide Rule, 24th Ed., Pitman, London, 1945, p. 53, Gauge Points.

LINKS G. C. Greubel, Table of n, a(n) for n = 0..10000

Mohammad K. Azarian, An Expression for Pi, Problem #870, College Mathematics Journal, Vol. 39, No. 1, January 2008, p. 66. Solution appeared in Vol. 40, No. 1, January 2009, pp. 62-64. - Mohammad K. Azarian, Feb 08 2009

J. Bohr, Ramanujan's Method of Approximating Pi

# Borweins' quote: Gauß and $1.85407467\ldots$

THE ON-LINE ENCYCLOPEDIA OF INTEGER SEQUENCES ®

founded in 1964 by N. J. A. Sloane

1,8,5,4,0,7,4,6,7 Suche Tips

(Grüße von der Online-Enzyklopädie der Zahlenfolgen!)

Suche: **seq:1,8,5,4,0,7,4,6,7**

Displaying 1-1 of 1 result found. page 1

Sort: relevance | references | number | modified | created Format: long | short | data

A093341 Decimal expansion of "lemniscate case". +30 12

**1, 8, 5, 4, 0, 7, 4, 6, 7**, 7, 3, 0, 1, 3, 7, 1, 9, 1, 8, 4, 3, 3, 8, 5, 0, 3, 4, 7, 1, 9, 5, 2, 6, 0, 0, 4, 6, 2, 1, 7, 5, 9, 8, 8, 2, 3, 5, 2, 1, 7, 6, 6, 9, 0, 5, 5, 8, 5, 9, 2, 8, 0, 4, 5, 0, 5, 6, 0, 2, 1, 7, 7, 6, 8, 3, 8, 1, 1, 9, 9, 7, 8, 3, 5, 7, 2, 7, 1, 8, 6, 1, 6, 5, 0, 3, 7, 1, 8, 9, 7, 2, 7, 7, 7, 7 (list; constant; graph; refs; listen; history; text; internal format)

OFFSET 1,2

REFERENCES
M. Abramowitz and I. A. Stegun, eds., Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables, 9th printing. New York: Dover, 1972, Section 18.14.7, p. 658.
Jonathan Borwein & Peter Borwein, A Dictionary of Real Numbers. Pacific Grove, California: Wadsworth & Brooks/Cole Advanced Books & Software, 1990, p. iii.
Steven R. Finch, Mathematical Constants, Cambridge University Press, 2003, Section 6.1 Gauss' Lemniscate Constant, p. 421.

LINKS
Harry J. Smith, Table of n, a(n) for n = 1..5000
M. Abramowitz and I. A. Stegun, eds., Handbook of Mathematical Functions, National Bureau of Standards, Applied Math. Series 55, Tenth Printing, 1972 [alternative scanned copy].
M. Abramowitz and I. A. Stegun, eds., Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables, Section 18.14.7, p. 658.
G. Mingari Scarpello, D. Ritelli, On computing some special values of hypergeometric functions, arXiv:1212.0251 [math.CA], 2012-2014, eq. (4.1).
Eric Weisstein's World of Mathematics, Lemniscate Case.

FORMULA
GAMMA(1/4)^2/(4*(Pi)^(1/2)). - Pab Ter (pabrlos(AT)yahoo.com), May 24 2004
Also equals ellipticK(1/sqrt(2)) = Pi/2*hypergeom([1/2,1/2],[1],1/2),
or also the smallest positive root of cs(x/sqrt(2)|-1), where cs is the Jacobi elliptic function, or also the real half-period of the Weierstrass Pe function (Cf. Finch p. 422). - Jean-François Alcover, Apr 30 2013, updated Aug 01 2014
From Peter Bala, Feb 22 2015: (Start)
Equals Integral_{x = 0..oo} 1/sqrt(1 + x^4) dx = 2 * Integral_{x = 0..1} 1/sqrt(1 + x^4) dx = sqrt(2) * Integral_{x = 0..1} 1/sqrt(1 - x^4) dx.
Equals 2 * Sum {n >= 0} (-1/4)^n * binomial(2*n,n) * 1/(4*n + 1). (End)
Equals A062539 / sqrt(2). - Amiram Eldar, May 04 2022

EXAMPLE 1.854074677301371918433850347195260046217598823521766905585928045056021...

MAPLE evalf( EllipticK(1/sqrt(2)) ); # R. J. Mathar, Aug 28 2013

MATHEMATICA RealDigits[ N[ Gamma[1/4]^2 / (4*Sqrt[Pi]), 105]][[1]] (* Jean-François Alcover,

## Arithmetic-geometric mean (AGM) iteration & $1.85407467$

In[34]:= $\mathbf{a[n\_] := \frac{a[n-1]+b[n-1]}{2}; b[n\_] := \sqrt{a[n-1]b[n-1]};}$

## Arithmetic-geometric mean (AGM) iteration & $1.85407467$

In[39]:= $\mathbf{a[n\_] := \frac{a[n-1] + b[n-1]}{2}; b[n\_] := \sqrt{a[n-1]b[n-1]};}$

In[40]:= $\mathbf{\{a[0], b[0]\} = \{1, \sqrt{2}\};}$

In[41]:= $\mathbf{\{a[1], b[1]\} \quad //N[\#, 10]\&}$

## Arithmetic-geometric mean (AGM) iteration & $1.85407467$

In[44]:= $\mathbf{a[n\_] := \dfrac{a[n-1]+b[n-1]}{2}; b[n\_] := \sqrt{a[n-1]b[n-1]};}$

In[45]:= $\mathbf{\{a[0], b[0]\} = \{1, \sqrt{2}\};}$

In[46]:= $\mathbf{\{a[1], b[1]\} \quad //N[\#, 10]\&}$

Out[46]= $\{1.207106781, 1.189207115\}$

# Arithmetic-geometric mean (AGM) iteration & $1.85407467$

In[49]:= $\mathbf{a[n\_] := \frac{a[n-1]+b[n-1]}{2}; b[n\_] := \sqrt{a[n-1]b[n-1]};}$

In[50]:= $\mathbf{\{a[0], b[0]\} = \{1, \sqrt{2}\};}$

In[51]:= $\mathbf{\{a[1], b[1]\} \quad //N[\#, 10]\&}$

Out[51]= $\{1.207106781, 1.189207115\}$

In[52]:= $\mathbf{\{a[3], b[3]\} \quad //N[\#, 10]\&}$

# Arithmetic-geometric mean (AGM) iteration & 1.85407467

In[54]:= $\mathbf{a[n\_] := \dfrac{a[n-1]+b[n-1]}{2}; b[n\_] := \sqrt{a[n-1]b[n-1]};}$

In[55]:= $\mathbf{\{a[0], b[0]\} = \{1, \sqrt{2}\};}$

In[56]:= $\mathbf{\{a[1], b[1]\} \quad //N[\#, 10]\&}$

Out[56]= $\{1.207106781, 1.189207115\}$

In[57]:= $\mathbf{\{a[3], b[3]\} \quad //N[\#, 10]\&}$

Out[57]= $\{1.198140235, 1.198140235\}$

# Arithmetic-geometric mean (AGM) iteration & 1.85407467

In[59]:= $\mathbf{a[n\_] := \frac{a[n-1] + b[n-1]}{2}; b[n\_] := \sqrt{a[n-1]b[n-1]};}$

In[60]:= $\mathbf{\{a[0], b[0]\} = \{1, \sqrt{2}\};}$

In[61]:= $\mathbf{\{a[1], b[1]\} \quad //N[\#, 10]\&}$

Out[61]= $\{1.207106781, 1.189207115\}$

In[62]:= $\mathbf{\{a[3], b[3]\} \quad //N[\#, 10]\&}$

Out[62]= $\{1.198140235, 1.198140235\}$

---

In[63]:= $\mathbf{\frac{\pi}{\sqrt{2}} * \frac{1}{a[3]} \quad //N[\#, 10]\&}$

# Arithmetic-geometric mean (AGM) iteration & 1.85407467

In[64]:= $\mathbf{a[n\_] := \frac{a[n-1]+b[n-1]}{2}; b[n\_] := \sqrt{a[n-1]b[n-1]};}$

In[65]:= $\mathbf{\{a[0], b[0]\} = \{1, \sqrt{2}\};}$

In[66]:= $\mathbf{\{a[1], b[1]\} \quad //N[\#, 10]\&}$

Out[66]= $\{1.207106781, 1.189207115\}$

In[67]:= $\mathbf{\{a[3], b[3]\} \quad //N[\#, 10]\&}$

Out[67]= $\{1.198140235, 1.198140235\}$

---

In[68]:= $\mathbf{\frac{\pi}{\sqrt{2}} * \frac{1}{a[3]} \quad //N[\#, 10]\&}$

Out[68]= 1.854074677

## AGM iteration & Gauß' diary (May 30, 1799)

This result "will surely open up a whole new field of analysis."

$$\frac{\pi}{\sqrt{2}} \cdot \frac{1}{\mathrm{AGM}(1, \sqrt{2})} = 1.85407467\ldots = \sqrt{2} \int_0^1 \frac{dt}{\sqrt{1 - t^4}}.$$

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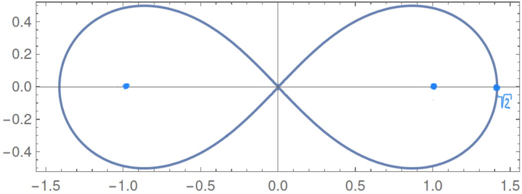


arc length of

$$(x^2+y^2)^2 = 2c^2(x^2-y^2)$$

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Gauß' ingenious extension:

$$\frac{1}{\mathrm{AGM}(1,\sqrt{1-k^2})} = \frac{2}{\pi}\underbrace{\int_0^{\pi/2} \frac{dx}{\sqrt{1-k^2\sin(x)^2}}}_{=:K(k)}.$$

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NOTE.

$$c \cdot \mathrm{AGM}(a,b) = \mathrm{AGM}(a\cdot c, b\cdot c).$$

$$\frac{1}{\mathrm{AGM}(1, \sqrt{1-k^2})} = \frac{2}{\pi} \underbrace{\int_0^{\pi/2} \frac{dx}{\sqrt{1-k^2 \sin(x)^2}}}_{=:K(k)}.$$

Complete elliptic integral of the 1st kind:

$$\begin{aligned} K(k) &= \frac{\pi}{2} \sum_{n=0}^{\infty} \left( \frac{(2n)!}{4^n n!} \right)^2 k^{2n} \\ &= \frac{\pi}{2} \sum_{n=0}^{\infty} \frac{(1/2)_n (1/2)_n}{(1)_n n!} k^{2n} \\ &= \frac{\pi}{2} {}_2F_1\left( \begin{matrix} \frac{1}{2} \; \frac{1}{2} \\ 1 \end{matrix} ; \, k^2 \right) \end{aligned}$$

$$K(k) = \frac{\pi}{2} {}_2F_1\left(\begin{matrix}\frac{1}{2} & \frac{1}{2} \\ & 1\end{matrix}; k^2\right)$$

Gauß connected this to Jacobi theta functions:

$$\boxed{\frac{\pi}{2}\theta_3(\tau)^2 = \frac{\pi}{2} {}_2F_1\left(\begin{matrix}\frac{1}{2} & \frac{1}{2} \\ & 1\end{matrix}; \underbrace{\left(\frac{\theta_2(\tau)^2}{\theta_3(\tau)^2}\right)^2}_{=:\lambda(\tau)}\right),}$$

$$K(k) = \frac{\pi}{2} {}_2F_1\left(\begin{matrix}\frac{1}{2} \ \frac{1}{2} \\ 1\end{matrix}; k^2\right)$$

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where

$$\theta_3(\tau) = \sum_{n\in\mathbb{Z}} x^{n^2} = 1 + 2x + 2x^4 + 2x^9 + 2x^{16} + 2x^{25} + \cdots$$

for $\boxed{x = x(\tau) = e^{i\pi\tau}}$, and $\theta_2(\tau)$ similar.

$$K(k) = \frac{\pi}{2}\,{}_2F_1\left(\begin{matrix}\frac{1}{2} & \frac{1}{2} \\ & 1\end{matrix};\, k^2\right)$$

Gauß connected this to Jacobi theta functions:

$$\frac{\pi}{2}\theta_3(\tau)^2 = \frac{\pi}{2}\,{}_2F_1\left(\begin{matrix}\frac{1}{2} & \frac{1}{2} \\ & 1\end{matrix};\, \underbrace{\left(\frac{\theta_2(\tau)^2}{\theta_3(\tau)^2}\right)^2}_{=:\lambda(\tau)}\right),$$

Modular lambda function:

$$\lambda\left(\frac{a\tau+b}{c\tau+d}\right) = \lambda(\tau) \text{ if } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod 2$$

## Remarks on arc length computations

$$1.85407467\ldots = K\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{2}\cdot\frac{1}{\mathrm{AGM}(1, 1/\sqrt{2})}$$

$$\stackrel{(1)}{=} \frac{\pi}{2}\,{}_2F_1\left(\begin{matrix}\frac{1}{2} & \frac{1}{2}\\ & 1\end{matrix}; k^2\right) \text{ with } k = \frac{1}{\sqrt{2}}$$

$$= \frac{\pi}{2}\,{}_2F_1\left(\begin{matrix}\frac{1}{2} & \frac{1}{2}\\ & 1\end{matrix}; \lambda(\tau)\right) \text{ with } \boxed{\lambda(\tau) = k^2} = \frac{1}{2}$$

$$= \frac{\pi}{2}\theta_3(\tau)^2 \stackrel{(2)}{=} \frac{\pi}{2}\left(1 + 2\sum_{n=1}^{\infty} x(\tau)^{n^2}\right)^2 \text{ with } x(\tau) = e^{i\pi\tau}$$

S. Cooper: One of the deepest results of elliptic functions is the

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S. Cooper: One of the deepest results of elliptic functions is the

INVERSION THEOREM. $\boxed{\tau = i\frac{K(1-k^2)}{K(k^2)}}$ solves $\boxed{\lambda(\tau) = k^2}$.

If $k = \frac{1}{\sqrt{2}}$: $\boxed{\tau = i\frac{K(1-k^2)}{K(k^2)} = i}$ solves $\boxed{\lambda(\tau) = k^2 = \frac{1}{2}}$

Consequently,

$$1.85407467\ldots = K\left(\frac{1}{\sqrt{2}}\right)$$

$$\overset{(1)}{=} \frac{\pi}{2}\,{}_2F_1\left(\begin{matrix}\frac{1}{2}\;\frac{1}{2}\\1\end{matrix};\,k^2\right) \text{ with } k = \frac{1}{\sqrt{2}}$$

$$\overset{(2)}{=} \frac{\pi}{2}\left(1 + 2\sum_{n=1}^{\infty} x(i)^{n^2}\right)^2 \text{ with } x(i) = e^{i\pi i} = e^{-\pi}$$

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$$\overset{(2)}{=} \frac{\pi}{2}\left(1 + 2\sum_{n=1}^{\infty} x(i)^{n^2}\right)^2 \text{ with } x(i) = e^{i\pi i} = e^{-\pi}$$

---

In[69]:= $\mathbf{S_1[n\_] := \frac{Pi}{2} Sum[\left(\frac{(2n)!}{4^n (n!)^2}\right)^2 \left(\frac{1}{2}\right)^n, \{n, 0, N\}]}$

In[70]:= $\mathbf{\{S_1[21], S_1[22]\} // N[\#, 10]\&}$

Out[70]= $\{1.854074667, 1.854074672\}$

If $k = \frac{1}{\sqrt{2}}$: $\boxed{\tau = i\dfrac{K(1-k^2)}{K(k^2)} = i}$ solves $\boxed{\lambda(\tau) = k^2 = \dfrac{1}{2}}$

Consequently,

$$1.85407467\ldots = K\left(\frac{1}{\sqrt{2}}\right)$$
$$\overset{(1)}{=} \frac{\pi}{2}\,{}_2F_1\left(\begin{matrix}\frac{1}{2} & \frac{1}{2} \\ & 1\end{matrix}; k^2\right) \quad \text{with} \quad k = \frac{1}{\sqrt{2}}$$
$$\overset{(2)}{=} \frac{\pi}{2}\left(1 + 2\sum_{n=1}^{\infty} x(i)^{n^2}\right)^2 \quad \text{with} \quad x(i) = e^{i\pi i} = e^{-\pi}$$

---

In[71]:= $\mathbf{S_2[n\_] := \frac{Pi}{2}\left(Sum[1 + 2\,Sum[Exp[-Pi \cdot n^2], \{n, 1, N\}]\right)^2}$

In[72]:= $\mathbf{\{S_2[1], S_2[2]\}//N[\#, 10]\&}$

Out[72]= $\{1.854050872, 1.854074677\}$

Identities such as

$$\frac{\pi}{2}\theta_3(\tau)^2 = \frac{\pi}{2}\,{}_2F_1\left(\begin{matrix}\frac{1}{2}\ \frac{1}{2}\\ 1\end{matrix};\underbrace{\left(\frac{\theta_2(\tau)^2}{\theta_3(\tau)^2}\right)^2}_{=:\lambda(\tau)}\right),$$

and also theorems such as the

INVERSION THEOREM $\boxed{\tau = i\frac{K(1-k^2)}{K(k^2)}}$ solves $\boxed{\lambda(\tau) = k^2}$,

can be discovered and proven ALGORITHMICALLY
using a **“First guess, then prove” strategy.** (Stay tuned . . . )

Identities such as

$$\frac{\pi}{2}\theta_3(\tau)^2 = \frac{\pi}{2}\,{}_2F_1\left(\begin{matrix}\frac{1}{2}\;\frac{1}{2}\\1\end{matrix};\underbrace{\left(\frac{\theta_2(\tau)^2}{\theta_3(\tau)^2}\right)^2}_{=:\lambda(\tau)}\right),$$

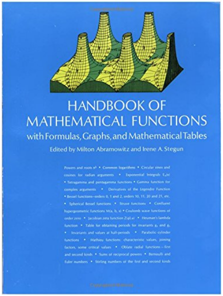
and also theorems such as the

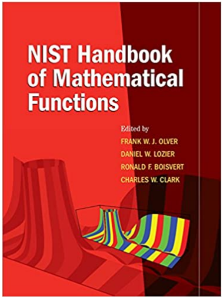
INVERSION THEOREM $\boxed{\tau = i\frac{K(1-k^2)}{K(k^2)}}$ solves $\boxed{\lambda(\tau) = k^2}$,

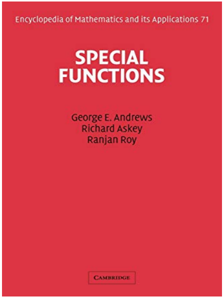
**Transformations: also relevant to MATCHING DATA!**

# Matching formulas, identities, theorems, etc

## Tables, etc.

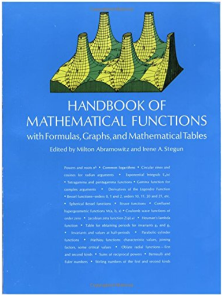


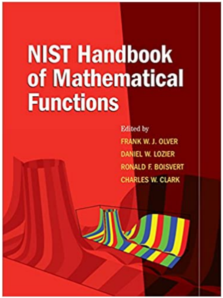



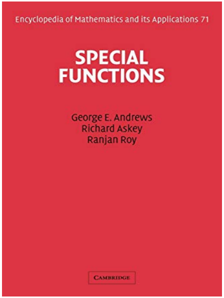



# Matching formulas, identities, theorems, etc

## Tables, etc.




Holonomic functions/sequences:
60 percent in the 1964 "Handbook" by Abramowitz and Stegun,
20 percent in Sloane's OEIS.

# Identifying patterns in data

# I.Q. Tests

In[73]:= << **RISC`GeneratingFunctions`**

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written by Christian Mallinger © RISC-JKU

# I.Q. Tests

In[78]:= << **RISC`GeneratingFunctions`**

Package GeneratingFunctions version 0.7
written by Christian Mallinger © RISC-JKU

In[79]:= **GuessNext2Values[Li_] := Module[{rec},**

**rec = GuessRE[Li, c[k], {1, 2}, {0, 3}];**

**RE2L[rec[[1]], c[k], Length[Li] + 1]]**

# I.Q. Tests

In[83]:= `<< RISC`GeneratingFunctions``

Package GeneratingFunctions version 0.7
written by Christian Mallinger © RISC-JKU

In[84]:=
```
GuessNext2Values[Li_] := Module[{rec},
  rec = GuessRE[Li, c[k], {1, 2}, {0, 3}];
  RE2L[rec[[1]], c[k], Length[Li] + 1]]
```

In[85]:=
```
GuessNext2Values[{1, 2, 4, 8}]
```

# I.Q. Tests

```
In[89]:= << RISC`GeneratingFunctions`
```

Package GeneratingFunctions version 0.7
written by Christian Mallinger © RISC-JKU

```
In[90]:= GuessNext2Values[Li_] := Module[{rec},
         rec = GuessRE[Li, c[k], {1, 2}, {0, 3}];
         RE2L[rec[[1]], c[k], Length[Li] + 1]]

In[91]:= GuessNext2Values[{1, 2, 4, 8}]

Out[91]=  {1, 2, 4, 8, 16, 32}
```

# I.Q. Tests

In[94]:= << RISC`GeneratingFunctions`

Package GeneratingFunctions version 0.7
written by Christian Mallinger © RISC-JKU

```
In[95]:= GuessNext2Values[Li_] := Module[{rec},
    rec = GuessRE[Li, c[k], {1, 2}, {0, 3}];
    RE2L[rec[[1]], c[k], Length[Li] + 1]]
```

```
In[96]:= GuessNext2Values[{1, 2, 4, 8}]
```

Out[96]= $\{1, 2, 4, 8, 16, 32\}$

```
In[97]:= GuessNext2Values[{1, 3, 6, 10, 15, 21}]
```

# I.Q. Tests

In[100]:= << RISC`GeneratingFunctions`

Package GeneratingFunctions version 0.7
written by Christian Mallinger © RISC-JKU

In[101]:= GuessNext2Values[Li_] := Module[{rec},
rec = GuessRE[Li, c[k], {1, 2}, {0, 3}];
RE2L[rec[[1]], c[k], Length[Li] + 1]]

In[102]:= GuessNext2Values[{1, 2, 4, 8}]

Out[102]= {1, 2, 4, 8, 16, 32}

In[103]:= GuessNext2Values[{1, 3, 6, 10, 15, 21}]

Out[103]= {1, 3, 6, 10, 15, 21, 28, 36}

# I.Q. Tests

In[105]:= << **RISC`GeneratingFunctions`**

Package GeneratingFunctions version 0.7
written by Christian Mallinger © RISC-JKU

In[106]:=
```
GuessNext2Values[Li_] := Module[{rec},
  rec = GuessRE[Li, c[k], {1, 2}, {0, 3}];
  RE2L[rec[[1]], c[k], Length[Li] + 1]]
```

In[107]:= **GuessNext2Values[{1, 2, 4, 8}]**

Out[107]= {1, 2, 4, 8, 16, 32}

In[108]:= **GuessNext2Values[{1, 3, 6, 10, 15, 21}]**

Out[108]= {1, 3, 6, 10, 15, 21, 28, 36}

In[109]:= **GuessNext2Values[{1, 1, 2, 6, 24, 120}]**

# I.Q. Tests

```
In[111]:= << RISC`GeneratingFunctions`
```

Package GeneratingFunctions version 0.7
written by Christian Mallinger © RISC-JKU

```
In[112]:= GuessNext2Values[Li_] := Module[{rec},
          rec = GuessRE[Li, c[k], {1, 2}, {0, 3}];
          RE2L[rec[[1]], c[k], Length[Li] + 1]]

In[113]:= GuessNext2Values[{1, 2, 4, 8}]

Out[113]= {1, 2, 4, 8, 16, 32}

In[114]:= GuessNext2Values[{1, 3, 6, 10, 15, 21}]

Out[114]= {1, 3, 6, 10, 15, 21, 28, 36}

In[115]:= GuessNext2Values[{1, 1, 2, 6, 24, 120}]

Out[115]= {1, 1, 2, 6, 24, 120, 720, 5040}
```

In[116]:= **GuessNext2Values[{1, 1, 2, 3, 5, 8}]**

In[118]:= $\mathbf{GuessNext2Values[\{1, 1, 2, 3, 5, 8\}]}$

Out[118]= $\{1, 1, 2, 3, 5, 8, 13, 21\}$

In[119]:= **GuessNext2Values**[{1, 1, 2, 3, 5, 8}]

Out[119]= {1, 1, 2, 3, 5, 8, 13, 21}

What is the underlying algorithmic guessing scheme?

```
In[120]:= GuessNext2Values[{1, 1, 2, 3, 5, 8}]

Out[120]= {1, 1, 2, 3, 5, 8, 13, 21}
```

What is the underlying algorithmic guessing scheme?

- What is the idea behind?

```
GuessRE[{1, 2, 4, 8, 16}, c[k]]

{{-2 c[k] + c[1 + k] == 0, c[0] == 1}, ogf}

GuessRE[{1, 3, 6, 10, 15, 21}, c[k]]

{{(-3 - k) c[k] + (1 + k) c[1 + k] == 0, c[0] == 1}, ogf}

GuessRE[{1, 1, 2, 6, 24, 120}, c[k]]

{{(-1 - k) c[k] + c[1 + k] == 0, c[0] == 1}, ogf}

GuessRE[{1, 1, 2, 3, 5, 8}, c[k]]

{{-c[k] - c[1 + k] + c[2 + k] == 0, c[0] == 1, c[1] == 1}, ogf}
```

# The Ramanujan Machine

https://www.ramanujanmachine.com

## The Ramanujan Machine

Using algorithms to discover new mathematics

WELCOME PUBLICATIONS RESULTS CODE PARTICIPATE ABO

## Welcome

Fundamental constants like e and π are ubiquitous in diverse fields of science, including physics, biology, chemistry, geometry, and abstract mathematics. Nevertheless, for centuries new mathematical formulas relating fundamental constants are scarce and are usually discovered sporadically by mathematical intuition or ingenuity.

$$\frac{4}{3\pi-8} = 3 - \cfrac{1\cdot 1}{6 - \cfrac{2\cdot 3}{9 - \cfrac{3\cdot 5}{12 - \cfrac{4\cdot 7}{15 - \dots}}}}$$

The Ramanujan Machine is a novel way to do mathematics by harnessing your computer power to make new discoveries. The Ramanujan Machine already discovered dozens of new conjectures.

$$\frac{e}{e-2} = 4 - \cfrac{1}{5 - \cfrac{2}{6 - \cfrac{3}{7 - \cfrac{4}{8 - \dots}}}}$$

Our algorithms search for new mathematical formulas. The community can suggest proofs for the conjectures or even propose or develop new algorithms. Any new conjecture, proof, or algorithm suggested will be named after you.

We hope that the Ramanujan Machine project will inspire future generations about mathematics and AI-driven science. Please join our

To date, the Ramanujan Machine focused on four algorithms, variants of Meet-In-The-Middle (MITM) algorithm, Gradient Descent (GD), the Berlekamp-Massey algorithm, and a proprietary search algorithm, based on a phenomenon of factorial reduction. The algorithms are described in details in our publications.

Summarizing, from the examples shown:

- algorithms play an essential role in the analysis and production of data;

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- symbolic algorithms (computer algebra) are important, also in combination with AI techniques.

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- symbolic algorithms (computer algebra) are important, also in combination with AI techniques.

---

Andrews: "The fact that they have improved the irrationality exponent for the Catalan constant from 0.554 to 0.567 reveals that they are able to make contributions to really hard problems [...] But the contributions made so far are not of the calibre that using Ramanujan's name would suggest. Calling this the Ramanujan Machine is over the top."

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Summarizing, from the examples shown:

- algorithms play an essential role in the analysis and production of data;
- algorithmic guessing scheme: (e.g., continued fractions of a certain type);
- symbolic algorithms (computer algebra) are important, also in combination with AI techniques.

---

Zeilberger: "Eventually, humans will be obsolete. And as the complexity of AI-generated mathematics grows, mathematicians will lose track of what computers are doing and will be able to understand the calculations only in broad outline."

To date, the Ramanujan Machine focused on four algorithms, variants of Meet-In-The-Middle (MITM) algorithm, Gradient Descent (GD), the Berlekamp-Massey algorithm, and a proprietary search algorithm, based on a phenomenon of factorial reduction. The algorithms are described in details in our publications.

Summarizing, from the examples shown:

- algorithms play an essential role in the analysis and production of data;
- algorithmic guessing scheme: (e.g., continued fractions of a certain type);
- symbolic algorithms (computer algebra) are important, also in combination with AI techniques.

---

In the rest of the talk we present further examples of computer algebra applications in connection with classical mathematics.

# Koutschan's HolonomicFunctions package

The implementation of Zeilberger's holonomic systems approach

```
In[1]:= << RISC`HolonomicFunctions`
```

HolonomicFunctions Package version 1.7.3 (21-Mar-2017)
written by Christoph Koutschan
Copyright Research Institute for Symbolic Computation (RISC),
Johannes Kepler University, Linz, Austria

```
--> Type  ?HolonomicFunctions  for help.
```

allows computer-assisted proofs and discovery of a large class of special functions identities.

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allows computer-assisted proofs and discovery of a large class of special functions identities.

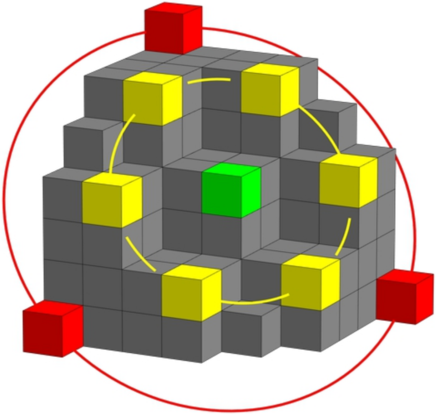
Zero recognition: compute generators of ideals of annihilating operators. Essential: $D$-finite (resp. holonomic) closure properties of function operations ($+$, $\times$, etc., incl. sums and integrals).

# AMS David P. Robbins Prize 2016

for paper in: Proceedings of the National Academy of Sciences (PNAS) **108(6)**, 2011;

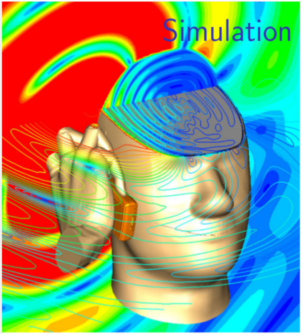
**Proof of George Andrews's and David Robbins's q-TSPP conjecture**

(by Christoph Koutschan, Manuel Kauers, Doron Zeilberger)



# Holonomic Surprise: a Patent!

## Simulation of electromagnetic waves



www.cst.com

- joint work by Joachim Schöberl (RWTH Aachen), Peter Paule and Christoph Koutschan (RISC)
- wide range of applications in constructing antennas, mobile phones, etc.
- merchandised by the company CST (Computer Simulation Technology)
- simulation with finite element methods
- significant contributions from Symbolic Computation using CK's package `HolonomicFunctions`
- symbolically derived formulae allow a considerable speed-up
- method is planned to be registered as a patent

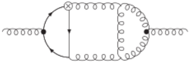
# Symbolic summation: Schneider's Sigma

## Symbolic Summation in Quantum Field Theory

RISC/JKU Collaboration with DESY (Berlin–Zeuthen)
(Deutsches Elektronen–Synchrotron)

| | |
|---|---|
| Project leader: | Carsten Schneider (RISC) |
| Partners: | Johannes Blümlein (DESY) |
| | Peter Paule (RISC) |

# Evaluation of Feynman Integrals



behavior of particles

$$\int \Phi(N, \epsilon, x)dx$$

Feynman integrals

**DESY**

LHC at CERN

applicable

$$\sum f(N, \epsilon, k)$$

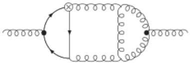
complicated multi-sums

**RISC**

(symbolic summation)

expression in special functions

# Evaluation of Feynman Integrals



Behavior of particles

$$\int \Phi(N, \epsilon, x) dx$$

Feynman integrals

**DESY**

LHC at CERN

applicable

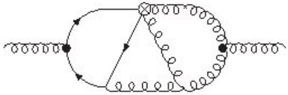
$$Dy = A\,y$$

coupled systems of linear DEs

**RISC**

(new coupled system solver)

expression in special functions



$$= F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + F_{-1}(N)\varepsilon^{-1} + \boxed{F_0(N)}$$

# Simplify

$$\|$$

$$\sum_{j=0}^{N-3}\sum_{k=0}^{j}\sum_{l=0}^{k}\sum_{q=0}^{-j+N-3}\sum_{s=1}^{-l+N-q-3}\sum_{r=0}^{-l+N-q-s-3}(-1)^{-j+k-l+N-q-3}\times$$

$$\times\frac{\binom{j+1}{k+1}\binom{k}{l}\binom{N-1}{j+2}\binom{-j+N-3}{q}\binom{-l+N-q-3}{s}\binom{-l+N-q-s-3}{r}r!(-l+N-q-r-s-3)!(s-1)}{(-l+N-q-2)!(-j+N-1)(N-q-r-s-2)(q+s+1)}$$

$$\Big[4S_1(-j+N-1)-4S_1(-j+N-2)-2S_1(k)$$

$$-(S_1(-l+N-q-2)+S_1(-l+N-q-r-s-3)-2S_1(r+s))$$

$$+2S_1(s-1)-2S_1(r+s)\Big]$$ + **3 further 6–fold sums**

# “First guess, then prove” strategies

**Example.** Recall $\boxed{x = e^{\pi i \tau}}$,

$$g(\tau) := \theta_3(\tau)^2 = 1 + 4x + 4x^2 + 4x^4 + 8x^5 + 4x^8 + 4x^9 + \dots$$

and

$$h(\tau) := \frac{\lambda(\tau)}{2^4} = \frac{1}{2^4}\frac{\theta_2(\tau)^4}{\theta_3(\tau)^4} = x(1 - 8x + 44x^2 + \dots).$$

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- $g(\tau)$ is a modular form of weight $1$ on $\mathbb{H}$; i.e.,

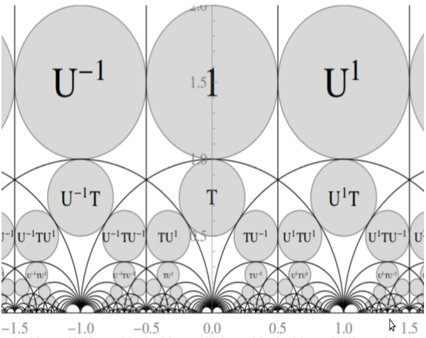
$$g\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^1 g(\tau) \text{ for certain } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

# VISUALIZING the SYMMETRY of MODULAR FUNCTIONS

$$j\left(\frac{a\tau+b}{c\tau+d}\right)=j(\tau) \text{ for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

---

NOTE. $j(\text{disk}_1)=j(\text{disk}_2)=\dots$ and $j(\tau)=\dfrac{256(1-\lambda(\tau)+\lambda(\tau)^2)^3}{\lambda(\tau)^2(1-\lambda(\tau)^2}$

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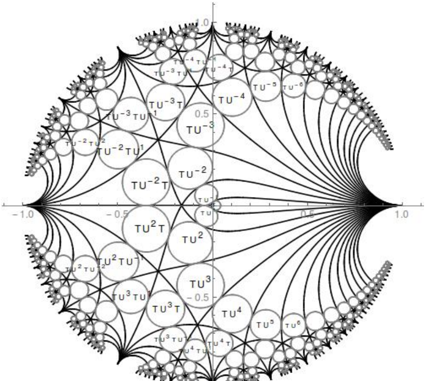

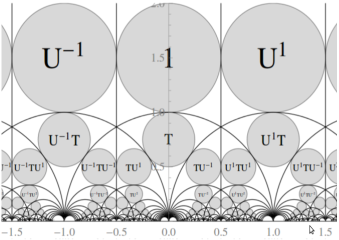


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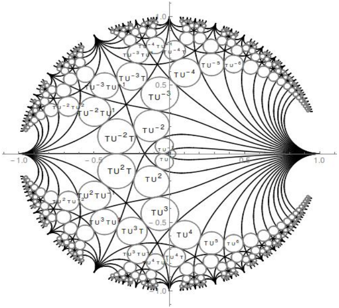
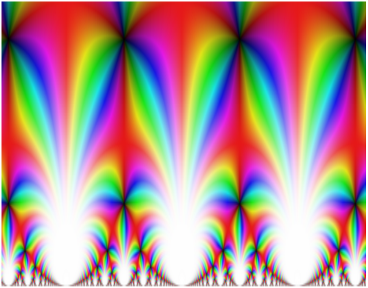
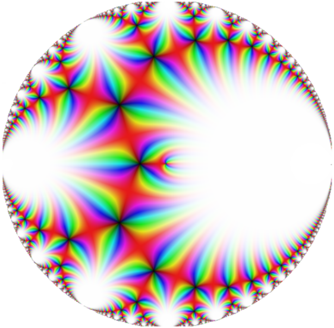


↔



From "Addition sequences . . ." by Fredrik Johansson, DK Statusseminar, Strobl 2015

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One can express $g(\tau)$ as a function in $h(\tau)$:

$$\exists y(z) := \sum_{n=0}^{\infty} c(n) z^n \text{ such that } \boxed{g(\tau) = y(h(\tau))}.$$

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- Modular forms are not holonomic, but $y(z)$ is!

$$\boxed{g(\tau) = y(h(\tau))} \Leftrightarrow \tilde{g}(x) = \sum_{n=0}^{\infty} c(n)\tilde{h}(x)^n$$

---

QUESTION. $g(\tau) = \theta_3(\tau)^2$ and $h(\tau) := \frac{\lambda(\tau)}{2^4}$, what are the $c(n)$?

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In[125]:= $\mathbf{\tilde{g} = 1 + 4x + 4x^2 + 4x^4 + 8x^5 + O[x]^6;}$

In[126]:= $\mathbf{\tilde{h} = x(1 - 8x + 44x^2 - 192x^3 + 718x^4 + O[x]^5);}$

In[127]:= $\mathbf{H = InverseSeries[\tilde{h}]}$

Out[127]= $x + 8x^2 + 84x^3 + 992x^4 + O[x]^5$

In[128]:= $\mathbf{ComposeSeries[g, H]}$

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I.e., we computed the first $c(n)$ as

$$\theta_3(\tau)^2 = \sum_{n=0}^{\infty} c(n)\left(\frac{\lambda(\tau)}{16}\right)^n = 1 + 4\frac{\lambda(\tau)}{16} + 36\frac{\lambda(\tau)^2}{16^2} + 400\frac{\lambda(\tau)^3}{16^3} + \dots,$$

Use the RISC package **GeneratingFunctions** to guess a recurrence for the $c(n)$:

In[133]:= $\mathbf{cList = \{1, 4, 36, 400, 4900, 63504, 853776, 11778624, 165636900, 2363904400, 34134779536, 497634306624\};}$

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$\Rightarrow$ CONJECTURE: We have

$$\theta_3(\tau)^2 = \sum_{n=0}^{\infty} c(n) \left(\frac{\lambda(\tau)}{16}\right)^n$$

for

$$c(n) = \frac{(1/2)_n (1/2)_n}{n!^2} 16^n = \left(\frac{(2n)!}{n!}\right)^2.$$

How to PROVE this?

PROOF of the CONJECTURE:

$$\theta_3(\tau)^2 = \sum_{n=0}^{\infty} \underbrace{\frac{(1/2)_n (1/2)_n}{n!^2} 16^n}_{=c(n)} \left(\frac{\lambda(\tau)}{16}\right)^n = \frac{\pi}{2} {}_2F_1\left(\begin{matrix}\frac{1}{2} & \frac{1}{2} \\ & 1\end{matrix}; \lambda(\tau)\right)$$

In[139]:= **cRec**

Out[139]= $\{-4(1+2n)^2 c[n] + (1+n)^2 c[1+n] = 0, c[0] = 1\}$

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1. Use **GeneratingFunctions** to convert cRec into a DE for $y(z)$:

In[142]:= **yDE = RE2DE[cRec, c[n], y[z]][[1]]**

PROOF of the CONJECTURE:

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1. Use **GeneratingFunctions** to convert cRec into a DE for $y(z)$:

In[144]:= **yDE = RE2DE[cRec, c[n], y[z]][[1]]**

Out[144]= $\{(16z^2 - z)y''[z] + (32z - 1)y'[z] + 4y[z] = 0, y[0] = 1, y'[0] = 4\}$

SUMMARY. For $y$ such that $\boxed{g(\tau) = y(h(\tau))}$ we obtained the

---

CONJECTURE:

If

$$(16h^2 - h)\frac{d^2y(h)}{dh^2} + (32h-1)\frac{dy(h)}{dh} + 4\,y(h) = 0 \text{ + initial values}$$

then

$$y(z) = \sum_{n=0}^{\infty} \underbrace{\frac{(1/2)_n(1/2)_n}{n!^2}16^n}_{=c(n)} z^n.$$

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---

NOTE. $g = y(h)$, $\frac{dy(h)}{dh} = \frac{g'}{h'}$, etc, hence the DE translates into:

$$\frac{h(-1+16h)}{(h')^2}g'' + \frac{hh'' - 16h^2h'' - (h')^2 + 32h(h')^2}{(h')^3}g' + 4g = 0.$$

We cannot proceed with this: e.g., coeffs. are not modular forms.

2. Convert

$$Y := (16h^2 - h)\frac{d^2y(h)}{dh^2} + (32h - 1)\frac{dy(h)}{dh} + 4\,y(h)$$

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Yifan Yang (2004): Any

$$Y := Q_m(h)\frac{d^m y(h)}{dh^m} + Q_{m-1}(h)\frac{d^{m-1}y(h)}{dh^{m-1}} + \cdots + Q_0(h)y(h),$$

with polynomials $Q_j(z) \in \mathbb{C}[z]$, can be written as

$$Y = \alpha_m \cdot g\frac{G_2^m}{G_1^m} + \alpha_{m-1} \cdot g\frac{G_2^{m-1}}{G_1^{m-1}} + \cdots + \alpha_0 \cdot g \ \text{ with } \ \alpha_j \in R,$$

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where $R$ is polynomial ring of modular functions,

$$R := \mathbb{C}\left[h, p_1, p_2, \frac{dp_1}{dh}, \frac{dp_2}{dh}, \ldots, \frac{d^mp_1}{dh^m}, \frac{d^mp_2}{dh^m}\right];$$

all these functions are constructed from $g, g'$ and $h, h'$.

**Example (cont'd.)** The Yang form for our DE is

$$Y = \alpha_1 \cdot g\frac{G_2}{G_1} + \alpha_0 \cdot g \text{ with } \alpha_1 := -p_1 h + p_1 + h$$

$$\alpha_0 := -p_2 h + p_2 + \frac{h}{4}.$$

Our algorithm provides the following bounds:

no. of poles$(\alpha_1) \leq 10$ and no. of poles$(\alpha_0) \leq 24$.

E.g., this means, to prove $\alpha_1 = 0$, it is sufficient to verify that the first $10$ coefficients in the $x$-expansion of $\alpha_1$ at $\infty$ are zero.

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- Peter Paule & Cristian-Silviu Radu: Holonomic relations for modular functions and forms: First guess, then prove, 2021.
- Peter Paule & Cristian-Silviu Radu: An Algorithm to Prove Holonomic Differential Equations for Modular Forms, 2021.


NOTE. The first article contains a variety of applications: Fricke-Klein relations, modular form based irrationality proofs by Beukers, etc.

## Ramanujan-Sato Series for $\frac{1}{\pi}$

- With our method one can discover and also prove, in a rigorous non-numerical manner, all of Ramanujan's $17$ series for $1$ over $\pi$, including (44) which Gosper used, and an identity of the Chudnovskys which is still used for world-record computations of $\pi$.

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- To this end, the "first guess, then prove" strategy is combined with a new algorithm MultiSamba to compute algebraic relations between modular functions.
- MultiSamba is an extension of Samba (subalgebra module basis algorithm). With MultiSamba one can find and prove evaluations of modular functions, at imaginary quadratic points, in terms of nested algebraic expressions.

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- Hemmecke, P. and Radu: Computer-assisted construction of Ramanujan-Sato series for $1$ over $\pi$, 2026.

## Ramanujan-Sato Series for $\frac{1}{\pi}$

- With our method one can discover and also prove, in a rigorous non-numerical manner, all of Ramanujan's 17 series for $1$ over $\pi$, including (44) which Gosper used, and an identity of the Chudnovskys which is still used for world-record computations of $\pi$.
- To this end, the "first guess, then prove" strategy is combined with a new algorithm MultiSamba to compute algebraic relations between modular functions.
- MultiSamba is an extension of Samba (subalgebra module basis algorithm). With MultiSamba one can find and prove evaluations of modular functions, at imaginary quadratic points, in terms of nested algebraic expressions.

---

MultiSamba is freely available as part of Ralf Hemmecke's package `QEta` written in FriCAS: `https://caa.risc.jku.at/software`

# Conclusion

## Another prominent example: Partition numbers $p(n)$

In[145]:= $\mathbf{Table[PartitionsP[n], n, 0, 14]}$

Out[145]= $\{1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, 101, 135\}$

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\prod_{j=1}^{\infty}\frac{(1-q^{5j})^5}{(1-q^j)^6}$$

$$\sum_{n=0}^{\infty} p(7n+5)q^n = 7\prod_{j=1}^{\infty}\frac{(1-q^{7j})^3}{(1-q^j)^4} + 49q\prod_{j=1}^{\infty}\frac{(1-q^{7j})^7}{(1-q^j)^8}$$

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NOTE 1. Given the left sides, **Radu's Ramanujan-Kolberg algorithm** computes both of Ramanujan's [1919] witness identities.

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NOTE 1. Given the left sides, **Radu's Ramanujan-Kolberg algorithm** computes both of Ramanujan's [1919] witness identities.

NOTE 2. Mathematica implementation of Radu's Ramanujan-Kolberg algorithm by Nicolas Smoot (RISC), implementation in FriCAS by Ralf Hemmecke (RISC).

I could present a collection of other examples of the unreasonable effectiveness of computer algebra, particularly in combinatorics, number theory and special functions, e.g.,

- Partition Analysis (MacMahon, Andrews, P. & Riese),
- Contiguous Relations (Gauß, Rainville, Gosper, P., etc.),
- Symbolic Summation ($q$-Zeilberger algorithm, etc.).

I could present a collection of other examples of the unreasonable effectiveness of computer algebra, particularly in combinatorics, number theory and special functions, e.g.,

- Partition Analysis (MacMahon, Andrews, P. & Riese),
- Contiguous Relations (Gauß, Rainville, Gosper, P., etc.),
- Symbolic Summation ($q$-Zeilberger algorithm, etc.).

I am sure that during the days of this conference we shall see plenty of other examples from other mathematical areas!

# Before MATCHING DATA: transform!

## Transformations: also relevant to MATCHING DATA!

"I recently had the occasion to look for a sequence in your book. It wasn't there. I tried the sequence of first differences. It *was* there and pointed me in the direction of the literature. Enchanting" (Herbert S. Wilf, University of Pennsylvania).

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Ex. Suppose you were led to consider

Out[148]= $1 + q + q^2 + q^3 + 2q^4 + 2q^5 + 3q^6 + 3q^7 + 4q^8 + 5q^9 + 6q^{10} + 7q^{11} + 9q^{12} + 10q^{13} + 12q^{14} + 14q^{15} + 17q^{16} + 19q^{17} + 23q^{18} + 26q^{19} + 31q^{20} + 35q^{21} + 41q^{22} + 46q^{23} + 54q^{24} + 61q^{25} + 70q^{26} + 79q^{27} + 91q^{28} + 102q^{29} + 117q^{30} + \dots$

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In[150]:= **bList30 = CoefficientList[%, q]**

Out[150]= $\{1, 1, 1, 2, 2, 3, 3, 4, 5, 6, 7, 9, 10, 12, 14, 17, 19, 23, 26, 31, 35, 41, 46, 54, 61, 70, 79, 91, 102, 117\}$

In[151]:= $\mathbf{bList30}$

Out[151]= $\{1, 1, 1, 2, 2, 3, 3, 4, 5, 6, 7, 9, 10, 12, 14, 17, 19, 23, 26, 31, 35, 41, 46, 54, 61, 70, 79, 91, 102, 117\}$

---

In[152]:= $\mathbf{b[n\_] := bList30[[n]]}$

In[153]:= $\mathbf{a[n\_] := a[n] = -b[n] - \frac{1}{n}Sum[d\, a[d], d, Drop[Divisors[n], -1]] - \frac{1}{n}Sum[b[n-j]Sum[d\, a[d], \{d, Divisors[j]\}], \{j, 1, n-1\}]}$

In[154]:= $\mathbf{aList = Map[a, Range[30]]}$

In[155]:= $\mathbf{bList30}$

Out[155]= $\{1, 1, 1, 2, 2, 3, 3, 4, 5, 6, 7, 9, 10, 12, 14, 17, 19, 23, 26, 31, 35, 41, 46, 54, 61, 70, 79, 91, 102, 117\}$

---

In[156]:= $\mathbf{b[n\_] := bList30[[n]]}$

In[157]:= $\mathbf{a[n\_] := a[n] = -b[n] - \frac{1}{n}Sum[d\, a[d], d, Drop[Divisors[n], -1]] - \frac{1}{n}Sum[b[n-j]Sum[d\, a[d], \{d, Divisors[j]\}], \{j, 1, n-1\}]}$

In[158]:= $\mathbf{aList = Map[a, Range[30]]}$

Out[158]= $\{-1, 0, 0, -1, 0, -1, 0, 0, -1, 0, -1, 0, 0, -1, 0, -1, 0, 0, -1, 0, -1, 0, 0, -1, 0, -1, 0, 0, -1, 0\}$

---

In[159]:= **bList30**

Out[159]= $\{1, 1, 1, 2, 2, 3, 3, 4, 5, 6, 7, 9, 10, 12, 14, 17, 19, 23, 26, 31, 35, 41, 46, 54, 61, 70, 79, 91, 102, 117\}$

---

In[160]:= **b[n_] := bList30[[n]]**

In[161]:= **a[n_] := a[n] = −b[n] − (1/n) Sum[d a[d], d, Drop[Divisors[n], −1]] − (1/n) Sum[b[n − j] Sum[d a[d], {d, Divisors[j]}], {j, 1, n − 1}]**

In[162]:= **aList = Map[a, Range[30]]**

Out[162]= $\{-1, 0, 0, -1, 0, -1, 0, 0, -1, 0, -1, 0, 0, -1, 0, -1, 0, 0, -1, 0, -1, 0, 0, -1, 0, -1, 0, 0, -1, 0\}$

---

TRANSFORM: $1+\sum_{n=1}^{\infty} b_n q^n = \prod_{n=1}^{\infty} (1-q^n)^{a_n}$

## REMARK on $1+\sum_{n=1}^{\infty} b_n q^n = \prod_{n=1}^{\infty} (1-q^n)^{a_n}$

In[163]:= Series $\left[1+\sum_{n=1}^{50} \frac{q^{n^2}}{\prod_{j=1}^{n}(1-q^j)}, \{q, 0, 30\}\right]$

Out[163]= $1+q+q^2+q^3+2q^4+2q^5+3q^6+3q^7+4q^8+5q^9+6q^{10}+7q^{11}+9q^{12}+10q^{13}+12q^{14}+14q^{15}+17q^{16}+19q^{17}+23q^{18}+26q^{19}+31q^{20}+35q^{21}+41q^{22}+46q^{23}+54q^{24}+61q^{25}+70q^{26}+79q^{27}+91q^{28}+102q^{29}+117q^{30}+O[q]^{31}$

In[164]:= Series $\left[\prod_{k=0}^{50} \frac{1}{(1-q^{5k+1})(1-q^{5k+4})}, \{q, 0, 30\}\right]$

REMARK on $1+\sum_{n=1}^{\infty} b_n q^n = \prod_{n=1}^{\infty} (1-q^n)^{a_n}$

---

In[165]:= Series $\left[1+\sum_{n=1}^{50} \frac{q^{n^2}}{\prod_{j=1}^{n}(1-q^j)}, \{q,0,30\}\right]$

Out[165]= $1+q+q^2+q^3+2q^4+2q^5+3q^6+3q^7+4q^8+5q^9+6q^{10}+7q^{11}+9q^{12}+10q^{13}+12q^{14}+14q^{15}+17q^{16}+19q^{17}+23q^{18}+26q^{19}+31q^{20}+35q^{21}+41q^{22}+46q^{23}+54q^{24}+61q^{25}+70q^{26}+79q^{27}+91q^{28}+102q^{29}+117q^{30}+O[q]^{31}$

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This suggests:

$$\boxed{1+\sum_{n=1}^{\infty}\frac{q^{n^2}}{(1-q)(1-q^2)\dots(1-q^n)}=\prod_{m=0}^{\infty}\frac{1}{(1-q^{5m+1})(1-q^{5m+4})}}$$

and, by an analogous computation,

$$\boxed{1+\sum_{n=1}^{\infty}\frac{q^{n^2}}{(1-q)(1-q^2)\dots(1-q^n)}=\prod_{m=0}^{\infty}\frac{1}{(1-q^{5m+1})(1-q^{5m+4})}}$$

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L.J. Rogers (1892), S. Ramanujan (1913), I. Schur (1917)



$$1+\sum_{n=1}^{\infty} \frac{q^{n^2}}{(1-q)(1-q^2)\dots(1-q^n)} = \prod_{m=0}^{\infty} \frac{1}{(1-q^{5m+1})(1-q^{5m+4})}$$

and, by an analogous computation,

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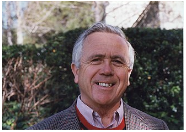
Rodney J. Baxter: hard hexagon model (1980) 

# Beukers' Irrationality proofs

# Irrationality proofs

- From Beukers' modular-form-based proof for the irrationality of $\zeta(3)$:

$$\prod_{n=1}^{\infty} \underbrace{\frac{(1-q^{2n})^7(1-q^{3n})^7}{1-q^n)^5(1-q^{6n})^5}}_{=g\in M_2(\Gamma_1(6))} = \sum_{k=0}^{\infty} c(k)\, \underbrace{q^k \prod_{n=1}^{\infty} \frac{(1-q^n)^{12k}(1-q^{6n})^{12k}}{1-q^{2n})^{12k}(1-q^{3n})^{12k}}}_{=h\in M(\Gamma_1(6))}.$$

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In[169]:= **cList20 = CoefficientList[Normal[Series[comp, {q, 0, 20}]], q];**

In[170]:= **AperyRec = GuessRE[cList20, c[k]][[1]]**

Out[170]= $\{(1+k)^3 c[k] - (3+2k)(39+51k+17k^2)c[1+k] + (2+k)^3 c[2+k] ==$
$0, c[0] == 1, c[1] == 5\}$

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In[171]:= **cList20 = CoefficientList[Normal[Series[comp, {q, 0, 20}]], q];**

In[172]:= **AperyRec = GuessRE[cList20, c[k]][[1]]**

Out[172]= $\{(1+k)^3 c[k] - (3+2k)(39+51k+17k^2) c[1+k] + (2+k)^3 c[2+k] = 0, c[0]=1, c[1]=5\}$

---

NOTE. Definite binomial sum representation of the $c(k)$:

$$c(k) := \sum_{\ell=0}^{k} \binom{k}{\ell}^2 \binom{k+\ell}{\ell}^2, \; k \geq 0.$$

# Partition Analysis

$$\sum_{\substack{a,b,c\geq 0\\ s.t.\ -6a+9b+20c=1}} x^a y^b z^c = ?$$

In[173]:= << **RISC‘Omega‘**

Omega Package version 2.49 written by Axel Riese (in cooperation with George E. Andrews and Peter Paule) © Research Institute for Symbolic Computation (RISC), Johannes Kepler University, Linz

$$\sum_{\substack{a,b,c\geq 0\\ s.t.\ -6a+9b+20c=1}} x^a y^b z^c = ?$$

---

In[176]:= $<<$ **RISC‘Omega‘**

> Omega Package version 2.49 written by Axel Riese (in cooperation with George E. Andrews and Peter Paule) © Research Institute for Symbolic Computation (RISC), Johannes Kepler University, Linz

In[177]:= $\mathbf{OEqSum[x^a y^b z^c, \{-6a + 9b + 20c == 1, a \geq 0, b \geq 0, c \geq 0\}, l]}$

$$\sum_{\substack{a,b,c\geq 0 \\ s.t.\ -6a+9b+20c=1}} x^a y^b z^c = ?$$

---

In[180]:= $<<$ **RISC‘Omega‘**

Omega Package version 2.49 written by Axel Riese (in cooperation with George E. Andrews and Peter Paule) © Research Institute for Symbolic Computation (RISC), Johannes Kepler University, Linz

In[181]:= $\mathbf{OEqSum[x^a y^b z^c, \{-6a + 9b + 20c == 1, a \geq 0, b \geq 0, c \geq 0\}, l]}$

Out[181]= $\underset{l_1}{\overset{\Omega}{=}} \dfrac{1}{\mathtt{l}_1 \left(1 - \frac{\mathtt{x}}{\mathtt{l}_1^6}\right) \left(1 - \mathtt{l}_1^9\, \mathtt{y}\right) \left(1 - \mathtt{l}_1^{20}\, \mathtt{z}\right)}$

In[182]:= $\mathbf{OEqR[\%[[1]], l_1]}$

$$\sum_{\substack{a,b,c\geq 0 \\ s.t.\ -6a+9b+20c=1}} x^a y^b z^c = ?$$

In[184]:= **<< RISC‘Omega‘**

Omega Package version 2.49 written by Axel Riese (in cooperation with George E. Andrews and Peter Paule) © Research Institute for Symbolic Computation (RISC), Johannes Kepler University, Linz

In[185]:= $\mathbf{OEqSum[x^a y^b z^c, \{-6a + 9b + 20c == 1, a \geq 0, b \geq 0, c \geq 0\}, l]}$

Out[185]= $\underset{l_1}{\overset{\Omega}{=}} \frac{1}{l_1 \left(1 - \frac{x}{l_1^6}\right)\left(1 - l_1^9\, y\right)\left(1 - l_1^{20}\, z\right)}$

In[186]:= $\mathbf{OEqR[\%[[1]], l_1]}$

Out[186]= $\frac{x^8 y z^2}{(1 - x^3 y^2)(1 - x^{10} z^3)}$

$$\sum_{\substack{a,b,c \geq 0 \\ s.t.\ -6a+9b+20c=1}} x^a y^b z^c = \frac{x^8 y^1 z^2}{(1-x^3y^2)(1-x^{10}z^3)};$$

in other words,

$$\begin{pmatrix} 8 \\ 1 \\ 2 \end{pmatrix} + \mathbb{N} \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} + \mathbb{N} \begin{pmatrix} 10 \\ 0 \\ 3 \end{pmatrix}$$

is the solution set of $-6a + 9b + 20c = 1$. For example,

$$8\,(-6) + 1\,(9) + 2\,(20) = 1.$$

# Inqualities

$$T(x,y,z;q) := \sum_{\substack{c \geq b \geq a \geq 1, \\ \text{s.t. } a+b>c}} x^a y^b z^c q^{a+b+c} = ?$$

# Inqualities

$$T(x,y,z;q) := \sum_{\substack{c \geq b \geq a \geq 1, \\ \text{s.t. } a+b>c}} x^a y^b z^c q^{a+b+c} = ?$$

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E.g., all triangles with given perimeter $n$ are given by the

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## Inqualities

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E.g., all triangles with given perimeter $n$ are given by the

coefficient of $q^n$ in $T(x,y,z;q)$,

and the total number of such triangles as the

coefficient of $q^n$ in $T(1,1,1;q)$.

---

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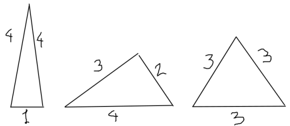
E.g., all triangles with given perimeter $n$ are given by the

coefficient of $q^n$ in $T(x,y,z;q)$,

and the total number of such triangles as the

coefficient of $q^n$ in $T(1,1,1;q)$.

EXAMPLE. $n = 9$



In[187]:= $\mathbf{OSum[x^a y^b z^c q^{a+b+c}, \{a + b > c, c \geq b, b \geq a\}, l]}$

In[191]:= $\mathbf{OSum[x^a y^b z^c q^{a+b+c}, \{a + b > c, c \geq b, b \geq a\}, l]}$

Out[191]= $$\underset{l_1, l_2, l_3}{\overset{\Omega}{\geq}} \frac{\mathtt{q\ x}}{\mathtt{l_1}\left(1 - \frac{\mathtt{l_1 q\ x}}{\mathtt{l_3}}\right)\left(1 - \frac{\mathtt{l_1 l_3 q\ y}}{\mathtt{l_2}}\right)\left(1 - \frac{\mathtt{l_2 q\ z}}{\mathtt{l_1}}\right)}$$

In[194]:= $\mathbf{OSum[x^a y^b z^c q^{a+b+c}, \{a + b > c, c \geq b, b \geq a\}, l]}$

Out[194]= $$\underset{l_1,l_2,l_3}{\overset{\Omega}{\geq}} \frac{\mathtt{q\ x}}{\mathtt{l_1}\left(1-\frac{\mathtt{l_1 q\ x}}{\mathtt{l_3}}\right)\left(1-\frac{\mathtt{l_1 l_3 q\ y}}{\mathtt{l_2}}\right)\left(1-\frac{\mathtt{l_2 q\ z}}{\mathtt{l_1}}\right)}$$

In[195]:= $\mathbf{OR[\%[[1]], \{l_1, l_2, l_3\}]}$

In[198]:= $\mathbf{OSum[x^a y^b z^c q^{a+b+c}, \{a + b > c, c \geq b, b \geq a\}, l]}$

Out[198]= $$\underset{l_1, l_2, l_3}{\overset{\Omega}{\geq}} \frac{\mathtt{q\ x}}{\mathtt{l_1}\left(1 - \frac{\mathtt{l_1 q\ x}}{\mathtt{l_3}}\right)\left(1 - \frac{\mathtt{l_1 l_3 q\ y}}{\mathtt{l_2}}\right)\left(1 - \frac{\mathtt{l_2 q\ z}}{\mathtt{l_1}}\right)}$$

In[199]:= $\mathbf{OR[\%[[1]], \{l_1, l_2, l_3\}]}$

Out[199]= $$\frac{q^3 xyz}{(1 - q^2 yz)(1 - q^3 xyz)(1 - q^4 xyz^2)}$$

In[201]:= $\mathbf{OSum[x^a y^b z^c q^{a+b+c}, \{a+b>c, c \geq b, b \geq a\}, l]}$

Out[201]= $$\underset{l_1,l_2,l_3}{\overset{\Omega}{\geq}} \frac{q\,x}{l_1\left(1-\frac{l_1 q\,x}{l_3}\right)\left(1-\frac{l_1 l_3 q\,y}{l_2}\right)\left(1-\frac{l_2 q\,z}{l_1}\right)}$$

In[202]:= $\mathbf{OR[\%[[1]], \{l_1, l_2, l_3\}]}$

Out[202]= $$\frac{q^3xyz}{(1-q^2yz)(1-q^3xyz)(1-q^4xyz^2)}$$

---

In[203]:= $\mathbf{Series[\%, q, 0, 9]}$

Out[203]= $$xyzq^3 + xy^2z^2q^5 + x^2y^2z^2q^6 + xyz(xyz^2 + y^2z^2)q^7 + x^2y^3z^3q^8$$
$$+ xyz(x^2y^2z^2 + xy^2z^3 + y^3z^3)q^9 + O[q]^{10}$$

---

In[204]:= $\mathbf{OSum[x^a y^b z^c q^{a+b+c}, \{a+b>c, c \geq b, b \geq a\}, l]}$

Out[204]= $$\underset{l_1, l_2, l_3}{\overset{\Omega}{\geq}} \frac{\mathtt{q\ x}}{\mathtt{l_1}\left(1-\frac{\mathtt{l_1 q\ x}}{\mathtt{l_3}}\right)\left(1-\frac{\mathtt{l_1 l_3 q\ y}}{\mathtt{l_2}}\right)\left(1-\frac{\mathtt{l_2 q\ z}}{\mathtt{l_1}}\right)}$$

In[205]:= $\mathbf{OR[\%[[1]], \{l_1, l_2, l_3\}]}$

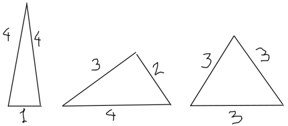
Out[205]= $$\frac{q^3xyz}{(1-q^2yz)(1-q^3xyz)(1-q^4xyz^2)}$$

---

In[206]:= $\mathbf{Series[\%, q, 0, 9]}$

Out[206]= $$xyzq^3 + xy^2z^2q^5 + x^2y^2z^2q^6 + xyz(xyz^2+y^2z^2)q^7 + x^2y^3z^3q^8 + xyz(x^2y^2z^2+xy^2z^3+y^3z^3)q^9 + O[q]^{10}$$

---



Most recent:

G.E. Andrews & Paule, MacMahon's Partition Analysis XIV : partitions with $n$ copies of $n$: RISC Report Series 22-14, 2022.

Many new results, including a partition analysis proof which made MacMahon's dream come true:

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Many new results, including a partition analysis proof which made MacMahon's dream come true:

$$\sum_{n=0}^{\infty} \mathrm{pp}(n) q^n = \frac{1}{1-q} \cdot \frac{1}{(1-q^2)^2} \cdot \frac{1}{(1-q^3)^3} \cdot \ldots$$

# Radu's Ramanujan-Kolberg algorithm

C.-S. Radu, An Algorithmic Approach to Ramanujan-Kolberg Identities, 2015

Smoot's Mathematica package:

In[207]:= << **RaduRK`**

math4ti2: Mathematica interface to 4ti2 (`http://www.4ti2.de`)
© 2017, Ralf Hemmecke `<ralf@hemmecke.org>`
© 2017, Silviu Radu `<sradu@risc.jku.at>`

RaduRK: Ramanujan–Kolberg Program Version 3.4 2021 written by Nicolas Smoot `<nicolas.smoot@risc.jku.at>` © Research Institute for Symbolic Computation (RISC), Johannes Kepler University Linz

Input: $(M, (r_\delta)_{\delta|M})$ to define the sequence $a(n), n \geq 0$,

$$\sum_{n=0}^{\infty} a(n)q^n := \prod_{j=1}^{\infty}(1-q^j)^{r_1} \cdot \prod_{j=1}^{\infty}(1-q^{2j})^{r_2} \cdot \prod_{j=1}^{\infty}(1-q^{3j})^{r_3} \cdots$$

$$= \prod_{\delta|M} \prod_{j=1}^{\infty}(1-q^{\delta j})^{r_\delta}$$

and $(m, j)$ to define the **generating function of interest**:

$$\sum_{n=0}^{\infty} a(m\ n + j)q^n.$$

---

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$$= \prod_{\delta|M} \prod_{j=1}^{\infty}(1-q^{\delta j})^{r_\delta}$$

and $(m, j)$ to define the **generating function of interest**:

$$\sum_{n=0}^{\infty} a(m\, n + j)q^n.$$

---

Ex. $(M, (r_\delta)_{\delta|M}) = (1, (-1))$ and $(m, j) = (5, 4)$ defines

$$\sum_{n=0}^{\infty} p(5n+4)q^n \quad \text{with} \quad \sum_{n=0}^{\infty} p(n)q^n = \prod_{j=1}^{\infty}(1-q^j)^{-1}.$$

---

Input: $(M,(r_\delta)_{\delta|M})$ to define the sequence $a(n), n\geq 0$,

$$\sum_{n=0}^{\infty} a(n)q^n := \prod_{j=1}^{\infty}(1-q^j)^{r_1}\cdot\prod_{j=1}^{\infty}(1-q^{2j})^{r_2}\cdot\prod_{j=1}^{\infty}(1-q^{3j})^{r_3}\cdots$$

$$=\prod_{\delta|M}\prod_{j=1}^{\infty}(1-q^{\delta j})^{r_\delta}$$

and $(m,j)$ to define the **generating function of interest**:

$$\sum_{n=0}^{\infty} a(m\, n+j)q^n.$$

---

Ex. $(M,(r_\delta)_{\delta|M})=(1,(-1))$ and $(m,j)=(5,4)$ defines

$$\sum_{n=0}^{\infty} p(5n+4)q^n \;\text{ with }\; \sum_{n=0}^{\infty} p(n)q^n=\prod_{j=1}^{\infty}(1-q^j)^{-1}.$$

---

In[210]:= $\mathbf{RK}[\mathbf{5},\mathbf{1},\{\mathbf{-1}\},\mathbf{5},\mathbf{4}]$

Out[125] =

| | |
|---|---|
| $N$: | $5$ |
| $\{M,(r_\delta)_{\delta\vert M}\}$: | $\{1,(-1)\}$ |
| $m$: | $5$ |
| $P_{m,r}(j)$: | $\{4\}$ |
| $f_1(q)$ : | $\frac{(q;q)_\infty^6}{(q^5;q^5)_\infty^5}$ |
| $t$: | $\frac{(q;q)_\infty^8}{q(q^4;q^4)_\infty^8}$ |
| AB: | $\{1\}$ |
| $\{p_g(t)$: $g\in$AB$\}$ | $\{5\}$ |
| Common Factor: | $5$ |

Out[125] =

| | |
|---|---|
| $N$: | $5$ |
| $\{M,(r_\delta)_{\delta\mid M}\}$: | $\{1,(-1)\}$ |
| $m$: | $5$ |
| $P_{m,r}(j)$: | $\{4\}$ |
| $f_1(q)$ : | $\frac{(q;q)_\infty^6}{(q^5;q^5)_\infty^5}$ |
| $t$: | $\frac{(q;q)_\infty^8}{q(q^4;q^4)_\infty^8}$ |
| AB: | $\{1\}$ |
| $\{p_g(t)$: $g\in$AB$\}$ | $\{5\}$ |
| Common Factor: | $5$ |

$$f_1(q)\sum_{n=0}^{\infty} p(5n+4)q^n = \sum_{g\in \mathrm{AB}} p_g(t)\cdot g = 5\cdot 1.$$

Out[125] =

| | |
|---|---|
| $N$: | $5$ |
| $\{M,(r_\delta)_{\delta\vert M}\}$: | $\{1,(-1)\}$ |
| $m$: | $5$ |
| $P_{m,r}(j)$: | $\{4\}$ |
| $f_1(q)$ : | $\frac{(q;q)_\infty^6}{(q^5;q^5)_\infty^5}$ |
| $t$: | $\frac{(q;q)_\infty^8}{q(q^4;q^4)_\infty^8}$ |
| AB: | $\{1\}$ |
| $\{p_g(t)\text{: } g \in \text{AB}\}$ | $\{5\}$ |
| Common Factor: | $5$ |

$$f_1(q)\sum_{n=0}^{\infty} p(5n+4)q^n = \sum_{g\in\text{AB}} p_g(t)\cdot g = 5\cdot 1.$$

$$\boxed{\sum_{n=0}^{\infty} p(5n+4)q^n = 5\prod_{j=1}^{\infty}\frac{(1-q^{5j})^5}{(1-q^j)^6}}$$